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ASYMPTOTIC DEVELOPMENT OF STEADY STATE ELECTROMAGNETIC FIELDS

by

RUDOLF K. LUNEBERG

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ABSTRACT

This paper develops two different forms of asymptotic series for the spatial behavior of steady state electromagnetic fields. The basic variable in each series is the wave-length so that the first few terms of each series may serve as excellent approximations to the steady state for small λ.

Each asymptotic expression for the steady state field is obtained in terms of what is called the pulse solution of Maxwell's equations. This is the field which is established by the sudden introduction of charge into the medium and which, with increasing time, approaches therefore the static field surrounding that charge. Actually the discontinuities of this field and of its derivatives with respect to the time variable determine the essential part of each coefficient of the terms in the asymptotic series.

The result permits linking the asymptotic series with the field furnished by geometrical optics. It is shown that the discontinuity in E and H which exists on any wave-front created by the sudden introduction of a charge or by reflection and refraction is precisely the value of E and H given by geometrical optics. It is shown also that the discontinuities in the E and H of the pulse solution at any fixed point in space at which ϵ, μ, and σ are continuous and at which there are no charges are discontinuities on wave fronts. Hence these discontinuities, which are given by geometrical optics, furnish the amplitude of first term of the asymptotic expansion of the steady state field.

1. Introduction*

This paper presents two forms of asymptotic expressions for the steady state electromagnetic field, arising from a periodic source, in the form of series whose basic variable is the wave-length. The series need not be convergent but can at least furnish an asymptotic representation of the field which is useful especially at small wave-lengths.

The full asymptotic representations are built up from a knowledge of what is called the pulse solution of Maxwell's equations. This solution is the field established by a source, i.e., charges, having the same geometrical distribution as the source of the steady state field being represented but having, instead of a periodic time behavior, the behavior of being 0 for $t < 0$ and 1 for $t \geq 0$. In the first of the asymptotic representations the first term is given essentially by the sum of the discontinuities in time of the pulse solution; the second term is given essentially by the sum of the discontinuities in time of the time derivatives of the pulse solution; the third term by the sum of the discontinuities in the second time derivatives of the pulse solution; etc. Of course each term of the representation contains the appropriate power of the wave-length, λ. This result is embodied in formula (9.18). Actually formula (9.18) gives an exact representation of the steady state field, but it leads directly to the asymptotic representation.

The second asymptotic representation, which replaces formula (9.18) when the discontinuities in the time derivatives of the pulse solution become infinite is obtained through the use of a series of complex integrals [formula (10.12)]. Each integral depends upon the singularities of the integrand which in turn depends upon the discontinuities of the pulse solution. Evaluation of these integrals leads to series in the variable λ.

Both asymptotic representations call for a knowledge of the discontinuities in the pulse solution and its derivatives. For the purposes of this paper it is assumed that the full pulse solution is known. This solution can be obtained, at least theoretically, by solution of Maxwell's equations. In an application of this theory it will be shown how the pulse solution is obtained in another manner for a class of diffraction problems.[1]

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* This paper was written up for Dr. R. K. Luneberg by Morris Kline. Crudities in the exposition are therefore to be charged to the writer.

[1] This application is now being written up as a report to the Geophysical Research Directorate of the Cambridge Field Station, U. S. Air Force.

The physical reason for the discontinuities in the pulse solution and its derivatives, at a point (x,y,z) in the case where ε, μ, and σ are continuous there and no sources are present, is the sudden passage through the point of the signal as it comes directly from the source and the subsequent passage of other signals which are refracted by obstacles elsewhere in the medium.

The subject of this paper must be distinguished from numerous special studies in electromagnetic theory wherein a given expression, representing the solution of a particular problem, is evaluated approximately by obtaining an asymptotic form of this expression valid for small wave lengths. For example, the diffraction of spherical waves by a half-plane has been so treated.[2] The asymptotic expression obtained is limited to the particular problem.

The general result of this paper makes possible a clarification of the relationship of geometrical optics to electromagnetic theory.[3] This relationship is established by means of a theoretical approach to geometrical optics which was developed by the author prior to this time.[4] The essence of this approach, described as the theory of wave-fronts, will be presented here for the sake of continuity in presentation and for convenient reference. In brief it says that the geometrical optics values of the electromagnetic vectors E and H at any point x,y,z are the values on the wave-front of the field created by the source, or, equally well, by the wave-front of the pulse solution arising from the source. Since the surfaces on which the discontinuities in the pulse solution are, at points where ε, μ, and σ are continuous and no charges are present, wave-fronts, it is apparent that the first term in the asymptotic expressions is essentially that given by geometrical optics.

The usefulness of the asymptotic theory of this paper for problems involving small wave lengths is apparent. The use of even the first two terms of the asymptotic expansion will furnish a better approximation than that furnished by geometrical optics alone. It is hoped that the theory herein presented will be applicable to many problems for which the steady state solution is either difficult of impossible to obtain exactly.

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[2] See, for example, Baker, B.B. & Copson, E.T.: The Mathematical Theory of Huygens' Principle, Oxford Univ. Press, 1939, pp. 95-98.

[3] A limited discussion of this relationship may be found in Baker & Copson, loc. cit., p. 79. See also Born, M.: Optik, Julius Springer, Berlin, 1933, pp.45-47.

[4] Luneberg, R.K.: The Mathematical Theory of Optics, Brown University Lectures, 1944, published in mimeographed form by Brown University, Providence, R.I. The germ of this theory of wave-fronts is also in Born, loc.cit. See also Silver, Samuel: Microwave Antenna Theory and Design, McGraw-Hill Book Co., N.Y., 1949, Chapter 4.

2. Basic Facts about the Electromagnetic Field.

2.1 For isotropic media Maxwell's equations are written

$$(2.1)\qquad \begin{aligned} &\operatorname{curl} H - \frac{\epsilon}{c} E_t - \frac{4\pi}{c}\sigma E = 0 \\ &\operatorname{curl} E + \frac{\mu}{c} H_t = 0 \end{aligned}$$

where ϵ, μ, σ are given functions of x, y, and z. We assume that these electromagnetic parameters are sectionally smooth functions of x, y, and z, that is that discontinuities in the functions and their first derivatives are finite jumps.

When sources are present in the field it is desirable to write these equations in slightly more general form. Let us assume that electric currents are produced by an enforced motion of electric charge. This means mathematically that the current field consists of two parts, the induced part, which is given by σE, and the enforced part. The latter is the current due directly to the motion of the charges of the source, which may be at a point or distributed in space. We represent the enforced current by $\frac{1}{4\pi} F_t(x,y,z,t)$ where $\vec{F}(x,y,z,t)$ is a given vector field determined by the source distribution. It will generally be assumed that the components of F are sectionally smooth. When the source is taken into account we must consider instead of the homogeneous equations (2.1) the inhomogeneous equations

$$(2.2)\qquad \begin{aligned} &\operatorname{curl} H - \frac{\epsilon}{c} E_t - \frac{4\pi}{c}\sigma E = \frac{1}{c} F_t \\ &\operatorname{curl} E + \frac{\mu}{c} H_t = 0 . \end{aligned}$$

The problem we shall consider, which we shall refer to as the general radiation problem, is to find solutions $E(x,y,z,t)$ and $H(x,y,z,t)$ which, at $t = 0$, assume given initial values $E(x,y,z,0)$; $H(x,y,z,0)$.

In most applications we shall assume that F, E and H are identically zero for $t \leqq 0$. This corresponds to the physical situation that an electromagnetic source begins to operate at $t = 0$ and an electromagnetic field is established under its influence. If moreover the enforced currents are confined to a small neighborhood of the origin (o,o,o), then $F \equiv 0$ except at this point and the resulting electromagnetic fields represent the radiation from a single point source at $x = y = z = 0$.

In case $\sigma = 0$ it follows from the first of equations (2.2), by taking the divergence of both sides, and integrating, that

$$\operatorname{div} \epsilon E = - \operatorname{div} F. \tag{2.3}$$

Since $\operatorname{div} \epsilon E = \rho$, the charge distribution, we see that F also determines the charge distribution.

2.2. For many practical applications we can assume that the enforced current is given by a vector field F(x,y,z,t) which is periodic in time. We expect, for physical reasons, that an electromagnetic field is established with increasing time which is likewise periodic in time and has the same period as the source. This limit solution is called a steady state solution of Maxwell's equations. (Its establishment will be demonstrated mathematically from the solution of the above general radition problem.) Since any time periodic solution can be obtained by superposition of simple harmonic solutions, we may assume without loss of generality that

$$\vec{F} = \vec{c}(x,y,z) \cos \omega t + \vec{c}^*(x,y,z) \sin \omega t \tag{2.4}$$

where $\vec{c}$ and $\vec{c}^*$ are vectors independent of t.

The associated steady state solutions then have an analogous form

$$\begin{aligned} E &= a(x,y,z) \cos \omega t + a^*(x,y,z) \sin \omega t \\ H &= b(x,y,z) \cos \omega t + b^*(x,y,z) \sin \omega t \end{aligned} \tag{2.5}$$

where a, a*, b, b* are vectors independent of t.

The discussion of steady state solutions is greatly facilitated by interpreting the vectors F, E, H as real parts of complex vectors. In fact we may write

$$\begin{aligned} F &= R\left\{\vec{g}\, e^{-i\omega t}\right\} \\ E &= R\left\{\vec{u}\, e^{-i\omega t}\right\} \\ H &= R\left\{\vec{v}\, e^{-i\omega t}\right\} \end{aligned} \tag{2.6}$$

where $\vec{g}$, $\vec{u}$, $\vec{v}$ are complex vectors

$$\begin{aligned} \vec{g} &= \vec{c} + i\, \vec{c}^* \\ \vec{u} &= \vec{a} + i\, \vec{a}^* \\ \vec{v} &= \vec{b} + i\, \vec{b}^* \end{aligned} \tag{2.7}$$

u and v are called the *amplitude vectors* of the electromagnetic field E,H; $T = \frac{2\pi}{\omega}$ the *time period*, $\nu = \frac{1}{T} = \frac{\omega}{2\pi}$ the *frequency*, $\lambda = \frac{2\pi c}{\omega}$ the *wave length*.

We obtain conditions for the complex vectors $\vec{u}$, $\vec{v}$ by introducing into Maxwell's equations the complex vectors whose real parts determine F, E, and H by (2.6); this substitution yields the time free form of Maxwell's equations

$$\begin{aligned} \text{curl}\,\vec{v} + ik\,\epsilon^*\vec{u} &= -ik\vec{g} \\ \text{curl}\,\vec{u} - ik\mu\vec{v} &= 0 \end{aligned} \tag{2.8}$$

where $\epsilon^* = \epsilon + \frac{4\pi i}{\omega}\sigma$ is called the *complex dielectric constant* and where $k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$.

In those parts of the space which are not occupied by sources we have $\vec{g} = 0$. At such points we find, merely by taking the divergence of each equation, that the steady state solutions $\vec{u}$, $\vec{v}$ satisfy the conditions

$$\text{div}\,\epsilon^*\vec{u} = \text{div}\,\mu\vec{v} = 0. \tag{2.9}$$

2.3. Using the usual meanings for D and B, we introduce

$W_e = \frac{1}{8\pi}\,E \cdot D$, the electric energy per unit volume

$W_m = \frac{1}{8\pi}\,H \cdot B$, the magnetic energy per unit volume

$W = W_e + W_m$, the electromagnetic energy per unit volume

$S = \frac{c}{4\pi}\,E \times H$, the *radiation vector of Poynting*.

The time period of electromagnetic fields is in general so small that it is impossible to observe the actual oscillations of E and H. The same rapid oscillation is true for the energy density W and the flux vector S. We are however in the position to observe the average values

$$\tilde{W} = \frac{1}{8\pi T}\int_0^T (\epsilon E^2 + \mu H^2)\,dt \quad\text{and}\quad \tilde{S} = \frac{c}{4\pi T}\int_0^T (E \times H)\,dt, \tag{2.10}$$

where T is the period of oscillation.

Let us express these observable average values in terms of the complex vectors $\vec{u}$ and $\vec{v}$. We introduce

$$E = \frac{1}{2}(u\, e^{-i\omega t} + \bar{u}\, e^{i\omega t})$$

(2.11)

$$H = \frac{1}{2}(v\, e^{-i\omega t} + \bar{v}\, e^{i\omega t})$$

in formulas (2.10) and find readily that

$$\tilde{W} = \frac{1}{16\pi}(\epsilon\, |u|^2 + \mu |v|^2)$$

(2.12)

$$\tilde{S} = \frac{c}{16\pi}(u \times \bar{v} + \bar{u} \times v)$$

where $\bar{u}$ is the conjugate complex vector of u, that, $\bar{u} = \vec{a} - i\,\vec{a}^*$, $|u| = \sqrt{\vec{a}^2 + \vec{a}^{*2}}$, and $u \times \bar{v} = (\vec{a} + i\,\vec{a}^*) \times (\vec{b} - i\,\vec{b}^*) = \vec{a} \times \vec{b} + i\,\vec{a}^* \times \vec{b} - i\,\vec{a} \times \vec{b}^* + \vec{a}^* \times \vec{b}^*$.

2.4. The electric vector

(2.13) $$E = \vec{a}\cos\omega t + \vec{a}^*\sin\omega t = \frac{1}{2}(u\, e^{-i\omega t} + \bar{u}\, e^{+i\omega t})$$

describes a closed curve in space if t varies from 0 to T. The geometrical characteristics of this curve provide us with another observable characteristic of the electromagnetic field, its polarization.

We note first that the curve must be in the plane determined by the vectors a and a*. It thus must be normal to the vector $a \times a^*$ and hence normal to the vector $\frac{1}{2i}(u \times \bar{u})$. This result allows us to determine the position of the plane of the polarization curve directly from the complex vector u.

We show next that the polarization curve is an ellipse. From equation (2.13) it follows that

$$E_1^2 - \frac{1}{2}u_1\bar{u}_1 = \frac{1}{4}(u_1^2\, e^{-2i\omega t} + \bar{u}_1^2\, e^{2i\omega t})$$

$$E_2^2 - \frac{1}{2}u_2\bar{u}_2 = \frac{1}{4}(u_2^2\, e^{-2i\omega t} + \bar{u}_2^2\, e^{2i\omega t})$$

$$E_3^2 - \frac{1}{2}u_3\bar{u}_3 = \frac{1}{4}(u^2\, e^{-2i\omega t} + \bar{u}^2\, e^{2i\omega t}) .$$

These are three linear equations in the two unknowns $e^{2i\omega t}$ and $e^{-2i\omega t}$. A common solution is possible only if the determinant of the coefficients and

constant terms is 0 ; that is, if

$$(2.14)\qquad \begin{vmatrix} E_1^2 - \frac{1}{2}u_1\bar{u}_1 , & u_1^2 , & \bar{u}_1^2 \\ E_2^2 - \frac{1}{2}u_2\bar{u}_2 , & u_2^2 , & \bar{u}_2^2 \\ E_3^2 - \frac{1}{2}u_3\bar{u}_3 , & u_3^2 , & \bar{u}_3^2 \end{vmatrix} = 0,$$

i.e., if $E = (E_1, E_2, E_3)$ lies on a surface of second order. Since E describes a plane curve we conclude that the polarization curve must be a conic section. it must be an ellipse because it is closed.

The state of polarization is characterized by the length and the position of the main axes of the ellipse. We find these axes by the extreme values of the expression

$$(2.15)\qquad E^2 = \frac{1}{4}(u^2 e^{-2i\omega t} + \bar{u}^2 e^{2i\omega t} + 2u \cdot \bar{u}) .$$

By differentiation

$$0 = -u^2 e^{-2i\omega t} + \bar{u}^2 e^{2i\omega t}$$

and hence

$$(2.16)\qquad e^{2i\omega t} = \pm\sqrt{\frac{u^2}{\bar{u}^2}} .$$

The extreme vectors E then may be found by inserting the value (2.16) in expression (2.13). The lengths of these vectors follow from formula (2.15), namely,

$$(2.17)\qquad E^2 = \frac{1}{2}(u \cdot \bar{u} \mp \sqrt{u^2\bar{u}^2}\ ) .$$

The ratio

$$(2.18)\qquad K^2 = \frac{u \cdot \bar{u} - \sqrt{u^2\,\bar{u}^2}}{u \cdot \bar{u} + \sqrt{u^2\,\bar{u}^2}}$$

determines the ellipticity of the polarization. The polarization is circular if $K = 1$; i.e., if $u^2 = \bar{u}^2 = 0$. The polarization is linear if $K = 0$; i.e., if

$$u \cdot \bar{u} - \sqrt{u^2\,\bar{u}^2} = 0$$

or if $u \times \bar{u} = 0$. This is the case if u is proportional to a real vector. The ellipse of polarization, then, degenerates into a section of a straight line.

Since $u(x,y,z)$ varies with x,y,z we see that the state of polarization will in general be different at different points of the space. A wave may be linearly polarized at certain points in space and elliptically polarized at other points.

3. Discontinuity Conditions for signal and Medium Discontinuities.

3.1. Ordinarily the conditions which the vectors E, D, H, and B must satisfy at a surface where ϵ, μ, or σ changes abruptly are derived by a physical argument which is reasonably satisfactory. However, these conditions do not cover the case of discontinuities existing in the signal or field itself at points where ϵ, μ, and σ are continuous. Because we shall be interested in the mathematical conditions holding on a wave-front, that is, the outermost region of penetration of a signal, we shall consider signal discontinuities and in doing so will approach the whole question of discontinuities from a more mathematical standpoint. It will be seen that these considerations lead to the usual discontinuity or boundary conditions arising from a change in the character of the medium.

3.2. We derive these conditions as follows. Consider a function $A(x,y,z,t)$. Let G be a domain in the four-dimensional x,y,z,t space, and let the hyper-surface Γ be its boundary. We assume that $A(x,y,z,t)$ is continuous and has sectionally continuous derivatives. Let $N = (x_N, y_N, z_N, t_N)$ be a four-dimensional unit vector[5] which represents the outside normal on Γ. Then we have the relation

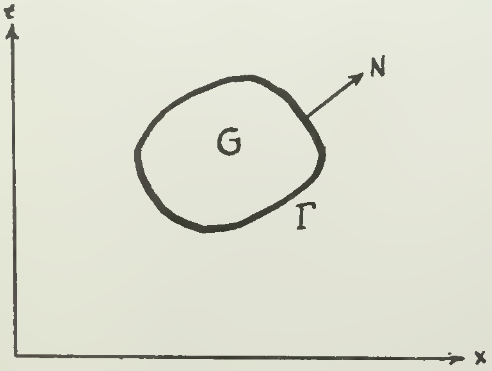

Fig. 1

$$\iiiint_G A_x \, dv = \iiint_\Gamma A \, x_N \, ds \tag{3.1}$$

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[5] If the equation defining Γ is $\phi(x,y,z,t) = 0$ then $x_N = \lambda \phi_x$, $y_N = \lambda \phi_y$, $z_N = \lambda \phi_z$, $t_N = \lambda \phi_t$ where $\lambda = \dfrac{1}{\pm\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2 + \phi_t^2}}$.

where dv = dx dy dz dt is the four-dimensional volume element and where ds is a three-dimensional element of the hyper-surface Γ. The analogous relations wherein x may be replaced by y, z, and t hold also.

The proof of this relationship in four-dimensional geometry is merely a generalization of the familiar fact that if we have an integral of any $f(x,y,z)$ over a volume, we may integrate partially with respect to z, say, substitute the z-boundaries in the integral and obtain two integrals over the projection of the original domain onto the xy-plane. However, the latter two integrals are equivalent to a surface integral over the boundary of the original volume in which integral the integrand must be multiplied by the direction cosine of the normal to the surface.

Consider next a vector field $\vec{A}(x,y,z,t)$. We find that

$$\begin{aligned}
\iiiint_G \operatorname{curl} \vec{A}\, dv &= \iiint_\Gamma (M \times \vec{A})\, ds \\
\iiiint_G \operatorname{div} \vec{A}\, dv &= \iiint_\Gamma (M \cdot \vec{A})\, ds \\
\iiiint_G \vec{A}_t\, dv &= \iiint_\Gamma \vec{A}\, t_N\, ds
\end{aligned} \tag{3.2}$$

where $M = (x_N, y_N, z_N, 0)$ is the projection of N onto the x, y, z space. In relations (3.2) curl, divergence, and cross product are used in the ordinary three-dimensional sense. To establish the relations we have to remember merely that curl and divergence are a sum of first order derivative terms and so we can apply relation (3.1) to each term and then recollect terms in accordance with the meaning of cross- and dot product.

Let us now assume that E, H, D, B and E are continuous and have sectionally continuous derivatives. If we denote the total current, namely $\sigma E + \frac{1}{4\pi} F_t$ by I we may write Maxwell's equations as

$$\begin{aligned}
\operatorname{curl} H - \frac{1}{c} D_t &= \frac{4\pi}{c} \sigma E + \frac{1}{c} F_t \qquad ; \operatorname{div} \left(I + \frac{1}{4\pi} D_t\right) = 0 \\
\operatorname{curl} E - \frac{1}{c} B_t &= 0 \qquad ; \operatorname{div} B = 0 \; .
\end{aligned} \tag{3.3}$$

We integrate these equations over an arbitrary four-dimensional domain G and make use of the relations (3.2). It follows that

$$\iiint_{\Gamma} (M \times H - \frac{t_N}{c} D - \frac{1}{c} t_N F)\, ds = \frac{4\pi}{c} \iiiint_{G} \sigma E \; dv \quad ; \iiint_{\Gamma} M \cdot (I + \frac{1}{4\pi} D_t)\, ds = 0$$

(3.4)

$$\iiint_{\Gamma} (M \times E + \frac{t_N}{c} B)\, ds = 0 \qquad ; \iiint_{\Gamma} M \cdot B \; ds = 0 .$$

These integral equations represent conditions for the vectors E, H, D. B. E and F. A derivative appears only in the second equation of (3.4). For continuous fields they are equivalent to the differential equations (3.3) insofar as we can retrace our steps and derive the equations (3.3). If, however, the field is discontinuous and satisfies the integral conditions (3.4) then the differential equations (3.3) will still be satisfied wherever the field is continuous. But in addition to this, certain conditions can be obtained by (3.4) which refer to the discontinuities of the field and which cannot be derived from the differential equations. Our next step will be to formulate these discontinuity conditions.

3.3. Let us assume that an electromagnetic field which satisfies the conditions (3.4), or the source, is discontinuous on a hypersurface

$$\phi(x,y,z,t) = 0.$$

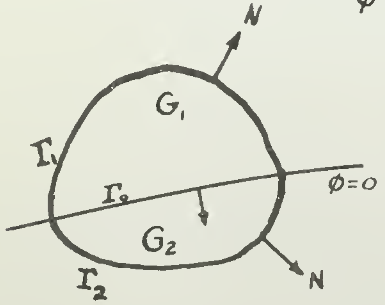

Fig. 2

We choose an arbitrary domain G which is divided into two parts G_1 and G_2 by $\phi = 0$. Also the boundary Γ of G is divided in two parts Γ_1 and Γ_2. Let Γ_0 be the part of $\phi = 0$ which lies inside G. We denote the discontinuous values of E, H, D, B, σE and F on Γ_0 by

$$E_1,\ H_1,\ D_1,\ B_1,\ \sigma_1 E_1,\ F_1$$

and

$$E_2,\ H_2,\ D_2,\ B_2,\ \sigma_2 E_2,\ F_2$$

and assume that the surface normal on Γ_o

$$\frac{1}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2 + \phi_t^2}} \quad (\phi_x, \phi_y, \phi_z, \phi_t)$$

points into the domain G_2.

We consider now the first equation (3.4) and apply it to the total domain G. It follows that

$$(3.5) \qquad \iiint_{\Gamma_1 + \Gamma_2} (M \times H - \frac{t_N}{c} D - \frac{1}{c} t_N F)\, ds = \frac{4\pi}{c} \iiiint_{G_1 + G_2} \sigma E \; dv.$$

However, the same condition must be satisfied by the domains G_1 and G_2 individually. This gives, using grad ϕ for the three-dimensional vector (ϕ_x, ϕ_y, ϕ_z),

$$\iiint_{\Gamma_1} (M x H - \frac{t_N}{c} D - \frac{1}{c} t_N F) \; ds \; + \iiint_{\Gamma_o} (\text{grad}\, \phi \, x \, H_1 - \frac{\phi_t}{c} D_1 - \frac{1}{c} \phi_t F_1) \frac{ds}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2 + \phi_t^2}}$$

$$(3.6) \qquad = \frac{4\pi}{c} \iiiint_{G_1} \sigma E \; dv$$

$$\iiint_{\Gamma_2} (M x H - \frac{t_N}{c} D - \frac{1}{c} F) \; ds - \iiint_{\Gamma_o} (\text{grad}\, \phi \, x H_2 - \frac{\phi_t}{c} - \frac{1}{c} \phi_t F_2) \frac{ds}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2 + \phi_t^2}}$$

$$= \frac{4\pi}{c} \iiiint_{G_2} \sigma E \; dv.$$

We add the equations (3.6) and subtract from (3.5). It follows

$$(3.7) \quad \iiint_{\Gamma_o} (\text{grad}\, \phi \, x (H_1 - H_2) - \frac{\phi_t}{c} (D_1 - D_2) - \frac{1}{c} \phi_t (F_1 - F_2)) \frac{ds}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2 + \phi_t^2}} = 0.$$

However, since Γ_o is an arbitrary hyper-area of $\phi = 0$ we conclude that the integrand in (3.7) must be zero and this gives

$$(3.8) \qquad \text{grad}\, \phi \, x [H] - \frac{\phi_t}{c} [D] - \frac{1}{c} \phi_t [F] = 0$$

where $H = H_1 - H_2$ and $D = D_1 - D_2$ are the jump discontimuities of H and D on $\Phi = 0$.

We apply the same consideration to the other equations (3.4) and thus find a system of difference equations:

$$\text{grad}\, \phi \times [H] - \frac{\phi_t}{c} [D] - \frac{1}{c} \phi_t [F] = 0 \qquad \text{grad}\, \phi \cdot \left\{ [I] + \frac{1}{4\pi} [D_t] \right\} = 0$$

(3.9)

$$\text{grad}\, \phi \times [E] + \frac{\phi_t}{c} [B] = 0 \qquad \text{grad}\, \phi \cdot [B] = 0$$

which establish conditions for discontimuities of an electromagnetic field. They are the counterpart of Maxwell's differential equations and can be obtained from them by replacing formally the differential operators

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}$$

by multiplicative operators

$$\phi_x, \phi_y, \phi_z, \phi_t .$$

3.4. A very special case of these discontinuity relations, which will be utilized later arises when the source F is discontimious at $t = 0$, so that the hypersurface $\phi \equiv t = 0$ is a surface of discontinuity of F, E and H. It follows from the first two of equations (3.9), since grad ϕ is now identically zero, that

$$(3.10) \qquad [\epsilon E + F] = 0 \quad \text{and} \quad [\mu H] = 0$$

and hence since $E = H = 0$ for $t < 0$ that

$$E(x,y,z,+0) = -\frac{1}{\epsilon} F(x,y,z,+0)$$

(3.11) and

$$H(x,y,z,+0) = 0$$

In case $\sigma = 0$ we conclude from Maxwell's equations (3.3) that $\frac{\partial}{\partial t}(\text{div}\, \epsilon E + F) = 0$ at any point where ϵE and F are continuous and have continuous derivatives. From (3.11) it follows that div $(\epsilon E +, F)$ approaches 0 if t approaches + 0 and hence div $\epsilon E = \rho = -$ div F in a certain neighborhood $t > 0$. However, it can readily be shown that this relation $\rho = -$ div F holds for all $t > 0$ as a consequence of the discontimity conditions (3.9).

3.5. The discontinuity relations (3.9) are general and hold regardless of the reason for the discontinuity in H, D. E, B, or F. We shall, for example, apply them to the surface of a wave-front. Discontinuous electromagnetic fields appear also on the boundaries of material bodies. In this case the material characteristics of the medium are discontinuous functions on surfaces $\psi(x,y,z) = 0$ which remain fixed in the three-dimensional space. In the four-dimensional x,y,z,t such a surface can be interpreted as a cylindrical hypersurface

$$\phi(x,y,z,t) = \psi(x,y,z) = 0$$

whose generating lines are parallel to the t axis. From (3.9) it follows that

$$\text{(3.12)} \qquad \begin{array}{ll} \operatorname{grad}\psi \times [H] = 0 & \operatorname{grad}\psi \cdot \left\{ [I] + \frac{1}{4\pi}[D_t] \right\} = 0 \\ \operatorname{grad}\psi \times [E] = 0 & \operatorname{grad}\psi \cdot [B] = 0 \end{array}$$

Since $\operatorname{grad}\psi \times E$ and $\operatorname{grad}\psi \times H$ are determined by the tangential components of E and H and $\operatorname{grad}\psi \cdot (I + \frac{1}{4\pi} D_t)$ and $B \cdot \operatorname{grad}\psi$ by the normal components of $I + \frac{1}{4\pi} D_t$ and B we can express (3.12) in the following usual manner.

The tangential components of E and H and the normal components of $I + \frac{1}{4\pi} D_t$ and of B are continuous on any boundary of material bodies.

4. Signal Discontinuities.

4.1. This next article will consider special relations which hold on a wave-front, that is, on the outermost surface reached by a signal at a given time t. The mathematical laws satisfied on a wave-front by the surface and by the electromagnetic vectors on the front constitute geometrical optics. While this concept of geometrical optics is not the familiar one, it will be seen that the laws of behavior of fronts and associated rays (to be considered in the next article) are precisely the familiar laws of geometrical optics. Our approach to geometrical optics will prove useful in the theory of steady state solutions to be discussed later.

4.2. Consider an electromagnetic field, at the time $t = 0$, which is different from zero only in a certain domain G_0 bounded by a surface Γ_0. The

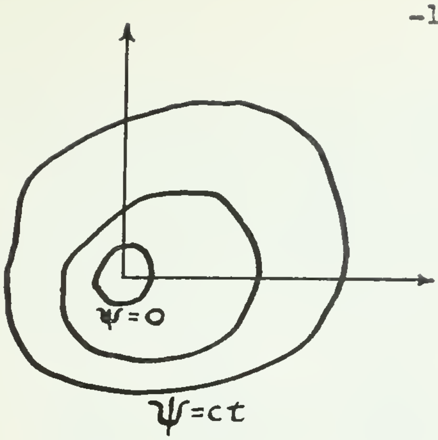

Fig. 3

electric excitation thus shall be confined to G_0. At a time $t > 0$ we expect the excitation to be expanded into a large domain G, bounded by a new surface Γ. Thus to any time t there belongs a surface $\phi(x,y,z,t) = 0$ which determines the boundary of the electromagnetic disturbance. Such surfaces are called Wavefronts. A set of wavefronts given by $\phi(x,y,z,t) = 0$ for different values of t in the x,y,z space generates in the four-dimensional x,y,z,t space a hyper-surface. Its "contour manifolds" t = const. give the wavefronts if projected onto the x,y,z space. If the electromagnetic vectors E and H are different from zero on the original surface then they will be found different from zero on all the later wavefronts. This means that they will be different from zero on the hypersurface $\phi = 0$. Since the field is zero outside this surface we have a field which is discontinuous on the hypersurface $\phi = 0$. The discontinuities $[E]$, $[H]$, $[D]$, $[B]$, $[I]$ moreover are equal to the vectors E, H, D, B, I themselves.

In this article we shall be concerned with these electromagnetic vectors on the wavefronts, i.e. with the electromagnetic field on the hypersurface $\phi = 0$. We shall find that these discontinuities establish a group of phenomena which can be treated separately insofar as the laws of propagation are independent of the field inside the wave surfaces, i.e. inside the hypersurface $\phi = 0$.

When $\phi_t \neq 0$ we may solve $\phi(x,y,z,t) = 0$ for t in the form

$$\phi = \psi(x,y,z) - ct = 0$$

so that the wavefront at the time t is given by the surface

$$\psi(x,y,z) = ct .$$

The function $\psi(x,y,z)$ is called the wave function. The electromagnetic vectors on the wave front then are given by the vector field.

$$E^* (x,y,z) = E(x,y,z, \frac{1}{c}\psi)$$

$$H^* (x,y,z) = H(x,y,z, \frac{1}{c}\psi) .$$

E* and H* thus give the first disturbance, i.e., the signal which is observed at a point x,y,z at the time $t = \frac{1}{c}\psi(x,y,z)$.

Our immediate problem will be to determine the laws satisfied by E* and H* and later to determine them from some initial information.

4.3. Let $\psi(x,y,z) = ct$ be a set of wavefronts obtained by varying t. We characterize the velocity v of propagation by the velocity of a point P of the wavefront on the associated orthogonal trajectory. Since ψ_x, ψ_y, and ψ_z are direction numbers of the normal to the surface at x,y,z, the velocity vector belonging to such a point is given by the vector

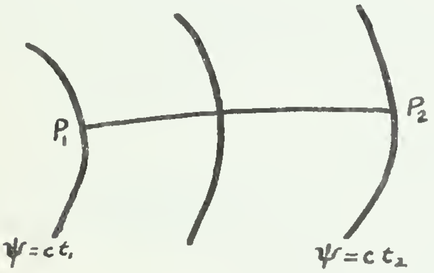

Fig. 4

$$(4.1)\qquad \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \frac{v}{\sqrt{\psi_x^2 + \psi_y^2 + \psi_z^2}}\,(\psi_x, \psi_y, \psi_z),$$

where v is the magnitude of the velocity of propagation and wherein x(t),y(t), and and z(t) are the parametric equation of the orthogonal trajectory. Since, on the other hand, (see Fig. 4),

$$\psi(P_2) - \psi(P_1) = c(t_2 - t_1)$$

if we regard $\psi(x,y,z)$ as a function of the x(t), y(t) and z(t) of an orthogonal trajectory then

$$(4.2)\qquad \frac{d\psi}{dt} = \psi_x \frac{dx}{dt} + \psi_y \frac{dy}{dt} + \psi_z \frac{dz}{dt} = c$$

By substituting for $\frac{dx}{dt}$, etc., from equation (4.1) we obtain the relation

$$(4.3)\qquad \sqrt{\psi_x^2 + \psi_y^2 + \psi_z^2} = \frac{c}{v}$$

or

$$(4.4)\qquad v = \frac{c}{\sqrt{\psi_x^2 + \psi_y^2 + \psi_z^2}}$$

The ratio

$$(4.5)\qquad n = \frac{c}{v}$$

is called the index of refraction and we have the general relation

$$\psi_x^2 + \psi_y^2 + \psi_z^2 = n^2 \quad . \tag{4.6}$$

Later we shall find that n is determined by the medium alone, independently of the particular electromagnetic excitation.

The vector

$$\vec{p} = \operatorname{grad} \psi \tag{4.7}$$

is called the wave normal. Its components are denoted by

$$p = \psi_x, \; q = \psi_y, \; r = \psi_z \; .$$

The index of refraction n thus is determined by the absolute value $|\vec{p}|$ of the wave normal $\vec{p}$.

If we differentiate the wave function $\psi(x,y,z)$ in a direction ρ which is not normal to the wavefronts we obtain from

$$\frac{d\psi}{d\rho} = p\frac{dx}{d\rho} + q\frac{dy}{d\rho} + r\frac{dz}{d\rho} = c\frac{dt}{d\rho} = \frac{c}{v} \tag{4.8}$$

the wave velocity v in the direction ρ. The ratio

$$\frac{c}{v} = N$$

is called the index of refraction in the direction ρ.

It is given by the scalar product

$$N = \vec{p} \cdot \vec{\rho} \tag{4.9}$$

where $\vec{\rho}$ is a unit vector in the desired direction and has the direction cosines

$$\frac{dx}{d\rho}, \; \frac{dy}{d\rho}, \; \frac{dz}{d\rho} \; .$$

4.4. The electromagnetic vectors E*, H*, D*, B* on the wavefronts $\phi = \psi(x,y,z) - ct = 0$ give directly the discontinuities [E], [H], [D], [B] of the field on the hypersurface $\phi = 0$. Consequently the equations (3.9) must be satisfied. If F is continuous on such a front then

$$\begin{aligned} \operatorname{grad}\phi \times H^* - \frac{\phi_t}{c} D^* &= 0 \\ \operatorname{grad}\phi \times E^* + \frac{\phi_t}{c} C^* &= 0 \quad . \end{aligned} \tag{4.10}$$

Since $\Phi = \psi - ct$ it follows that

$$\begin{aligned} &\text{grad}\,\psi \times H^* + D^* = 0 \\ (4.11)\qquad & \\ &\text{grad}\,\psi \times E^* - B^* = 0 \end{aligned}$$

We shall first assume that the medium itself is continuous, i.e. that the quantities ϵ, μ, and σ are continuous functions of x,y,z. Later on we shall supplement our results by allowing discontinuities also in these functions.

Under this assumption we may write

$$(4.12)\qquad \begin{aligned} \text{grad}\,\psi \times H^* + \epsilon E^* &= 0 \\ \text{grad}\,\psi \times E^* - \mu H^* &= 0 \end{aligned}$$

We conclude readily that

$$(4.13)\qquad E^* \cdot H^* = E^* \cdot \text{grad}\,\psi = H^* \cdot \text{grad}\,\psi = 0.$$

The vectors E*, H*, grad ψ from an orthogonal system of vectors; hence, the vectors E* and H* are tangential to the wavefront.

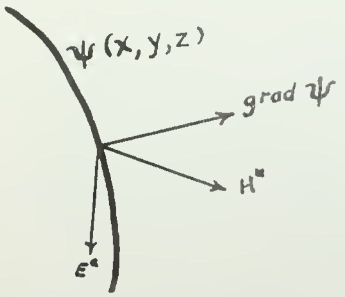

Fig. 5

The equations (4.12) are a homogeneous system of linear equations for the components of E* and H*. Since E* and H* are assumed to be different from zero we conclude that the determinant of (4.12) must be zero. This implies a condition which must be satisfied by the wavefront function $\psi(x,y,z)$. We derive this condition as follows. We eliminate H* from the first equation, obtaining

$$\text{grad}\,\psi \times (\text{grad}\,\psi \times E^*) + \epsilon\mu E^* = 0.$$

Since $E^* \cdot \text{grad}\,\psi = 0$ we get

$$(4.14)\qquad (\epsilon\mu - \text{grad}^2\,\psi)\,E^* = 0,$$

and hence

$$(4.15)\qquad \psi_x^2 + \psi_y^2 + \psi_z^2 = \epsilon\mu .$$

Wave fronts must be solutions of this partial differential equation which is known as the eiconal equation in geometrical optics.

We conclude, with the aid of (4.3):

The index of refraction, $n = \frac{c}{v}$, in an isotropic medium, is given by the relation

$$(4.16)\qquad n = \sqrt{\epsilon\mu} .$$

It is thus independent of the particular propagation and also independent of the direction of the waves.

4.5. We next multiply the first equation (4.12) with E* and the second with H*. We find

$$(4.17)\qquad (E^* \times H^*)\cdot \operatorname{grad}\Psi = \epsilon E^{*2}\,,\quad (E^*\times H^*)\cdot \operatorname{grad}\Psi = \mu H^{*2}\,,$$

which means that electric and magnetic energies are equal on a wave front. In formulae,

$$(4.18)\qquad \begin{aligned}\epsilon E^{*2} &= 4\pi W\\ \mu H^{*2} &= 4\pi W\end{aligned}$$

where W is the total energy density.

Poynting's radiation vector evaluated on the wave front, namely, $S^* = \frac{c}{4\pi}(E^*\times H^*)$ must be proportional to grad Ψ as follows from (4.13). From (4.17) and (4.18) it follows that

$$S^*\cdot \operatorname{grad}\Psi = cW$$

and thus

$$(4.19)\qquad S^* = \frac{c}{n^2}\, W \operatorname{grad}\Psi\,.$$

This gives finally by (4.15) that

$$(4.20)\qquad |S^*| = \frac{c}{n} W = v\, W\,,$$

i.e., a relation between flux and energy density, which is readily understood physically.

We note that the conductivity σ does not enter in our results. The shape of the wave fronts thus is influenced, also in metals, only by ϵ and μ.

5. Wave-Fronts and Rays

5.1. We have just surveyed the principle relationships satisfied by wave-fronts themselves as well as by the electric and magnetic vectors on the wave-fronts. We shall continue the study of wave-fronts and field vectors on the fronts by introducing a family of rays. A ray is by definition a curve which has at each point in space the direction of E*× H*. In isotropic media E*× H* has the direction of gradΨ where $\Psi(x,y,z) = ct$ are the wave-fronts (see formula (4.13)). Hence in isotropic media the rays are orthogonal trajectories of the wave-fronts.

After obtaining differential equations for the rays we shall study the behavior of the electric and magnetic field vectors on a front as they proceed along a particular ray and obtain differential equations for the electric and magnetic vectors along rays.

With the introduction of rays it becomes clearer that the theory of wave-fronts is equivalent to other approaches to geometric optics, for it is possible to show next that the rays satisfy familiar laws of optics such as Fermat's principle.[6] The next article will in fact establish Snell's law of reflection and refraction. Moreover it is possible to derive the mathematics of wave-fronts from a knowledge of the rays. Hence wave-fronts and rays are equivalent approaches.

We may repeat that the essence of what this and the few preceding articles establish is that the discontinuity relations and the eiconal equation are fundamental and that given a family of surfaces $\psi(x,y,z) = ct$ satisfying the eiconal equation and two vectors E* and H* satisfying the discontinuity relations (4.12) on this family we have geometrical optics studied as the theory of wave-fronts.

5.2. We have found in the preceding articles that electromagnetic wave-fronts $\psi - ct = 0$ must satisfy a partial differential equation of first order which for isotropic media is $\psi_x^2 + \psi_y^2 + \psi_z^2 = \epsilon\mu = n^2$. This equation is called the characteristic differential equation of Maxwell's equations. The constructions of wavefronts thus leads to the problem of integration of this differential equation. It is known from mathematical theory that the integration of partial differential equations of first order is equivalent to the integration of a certain system of ordinary differential equations, namely the characteristic differential equations of the first order partial differential equations. Its solutions are certain curves in space which are called the bicharacteristics of Maxwell's equations, or electromagnetic rays.

We introduce these curves here, not by mathematical but rather by physical considerations. We have seen in article 4.4. that on a system $\psi(x,y,z) = ct$ of wavefronts the radiation vector S* determines a vector field

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[6] See the Brown University Lectures referred to in footnote 3.

$$\vec{s}^* = \frac{E^* \times H^*}{4\pi W} \tag{5.1}$$

in space. A curve which at any of its points has the direction of the associated vector s* shall be called an electromagnetic ray.

Consider a set of wavefronts $\psi = ct$. We know that the radiation vector s* is normal to the wavefronts. The electromagnetic rays thus are the orthogonal trajectories of the set of wavefronts, i.e. a two parameter set of space curves which are normal to the surfaces $\psi = ct$. We represent these curves in parametric form $x = x(\sigma)$, $y = y(\sigma)$, $z = z(\sigma)$. Since they are normal to the surfaces they satisfy the differential relations

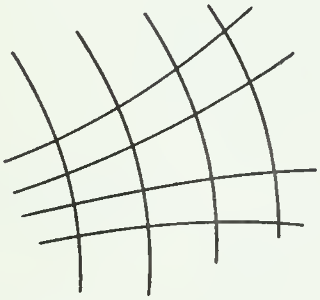

Fig. 6

$$\frac{dx}{d\sigma} = \lambda \psi_x, \quad \frac{dy}{d\sigma} = \lambda \psi_y, \quad \frac{dz}{d\sigma} = \lambda \psi_z \tag{5.2}$$

The factor λ can be chosen arbitrarily. These three equations form a system of ordinary differential equations for $x(\sigma)$, $y(\sigma)$, $z(\sigma)$. Their solutions give us certain space curves which are geometrically independent of the choice of λ. Indeed, the factor λ influences only the parametric representation of the curves and not the geometrical form. If another system of wavefronts $\psi = ct$ is considered, another two parameter set of rays is obtained. Thus it seems that any solution $\psi(x,y,z)$ of the eiconal equation (4.15) characterizes a set of rays and that the totality of all rays in a given medium will be a complicated set of curves of great variety. However, we shall see presently that this is not the case: The total set of rays in a given medium is determined by a four parameter set of space curves which can be characterized independently of the wave functions $\psi(x,y,z)$.

To prove this significant fact we divide (5.2) by λ and differentiate with regard to σ. The first equation yields

$$\frac{d}{d\sigma}\left(\frac{dx}{\lambda d\sigma}\right) = \psi_{xx}\frac{dx}{d\sigma} + \psi_{yx}\frac{dy}{d\sigma} + \psi_{zx}\frac{dz}{d\sigma}$$

$$= \lambda(\psi_{xx}\psi_x + \psi_{yx}\psi_y + \psi_{zx}\psi_z)$$

$$= \frac{\lambda}{2}\frac{\partial}{\partial x}(\psi_x^2 + \psi_y^2 + \psi_z^2) = \lambda\frac{\partial}{\partial x}\left(\frac{1}{2}n^2\right) .$$

The other equations give a similar result and thus a system of second order equations is formed:

$$(5.3)\qquad \begin{aligned} \frac{d}{\lambda d\sigma}\left(\frac{dx}{\lambda d\sigma}\right) &= \frac{\partial}{\partial x}\left(\frac{1}{2}n^2\right) \\ \frac{d}{\lambda d\sigma}\left(\frac{dy}{\lambda d\sigma}\right) &= \frac{\partial}{\partial y}\left(\frac{1}{2}n^2\right) \\ \frac{d}{\lambda d\sigma}\left(\frac{dz}{\lambda d\sigma}\right) &= \frac{\partial}{\partial z}\left(\frac{1}{2}n^2\right) \end{aligned}$$

in which the particular function $\psi(x,y,z)$ no longer appears. We conclude moreover from (5.2) that

$$(5.4)\qquad \left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right) = \lambda^2 n^2 .$$

Hence any electromagnetic ray $x(\sigma)$, $y(\sigma)$ $z(\sigma)$ must satisfy the differential equations (5.3) and the condition (5.4).

These functions $x(\sigma)$, $y(\sigma)$, $z(\sigma)$ depend on the choice of the factor λ. However, the integral curves which are represented by these functions do not. Indeed, a curve does not change its geometrical shape if the parameter σ is submitted to an arbitrary transformation. If now, on a particular curve for a certain parameter representation, the conditions (5.3) and (5.4) are satisfied then a new parameter σ' can be introduced such that the conditions (5.3) and (5.4) hold with $\lambda = 1$. These two equations are the basic equations for the rays and could be taken to be definitions of rays in an introduction to geometric optics which is independent of wave-fronts and based only on the physical properties of the medium contained in the quantity $n = \sqrt{\epsilon\mu}$.

5.3. Our discussion of the electromagnetic vectors, E^* and H^* thus far has not permitted us to determine them but just to note some properties. We shall now see how to determine the vectors E^* and H^* on a set of progressing wave-fronts $\psi = ct$.

In fact, it will be seen that these vectors are completely determined along a particular ray if they are known at one particular point of the ray.

The vectors $E = E^*$ and $H = H^*$ on the wave-fronts $\psi = ct$ are given by the electromagnetic vectors $E(x,y,z,t)$ and $H(x,y,z,t)$ if we insert $t = \frac{1}{c}\psi(x,y,z)$.

From

$$\begin{aligned} E^* &= E(x,y,z,\tfrac{1}{c}\psi) \\ H^* &= H(x,y,z,\tfrac{1}{c}\psi) \end{aligned} \tag{5.5}$$

it follows readily that[7]

$$\begin{aligned} \text{curl } E^* &= \text{curl } E + \frac{1}{c}\,\vec{p} \times E^*_t \\ \text{curl } H^* &= \text{curl } H + \frac{1}{c}\,\vec{p} \times H^*_t \ . \end{aligned} \tag{5.6}$$

Now by Maxwell's equations, which hold everywhere including the wave-fronts,

$$\begin{aligned} \text{curl } E &= -\frac{1}{c}\,\mu H_t \\ \text{curl } H &= \frac{1}{c}\,(\epsilon E_t + 4\pi\,\sigma E). \end{aligned}$$

We replace t by $\frac{\psi}{c}$ in these relations and then substitute in (5.6). Hence one obtains the relations:

$$\begin{aligned} \epsilon E^*_t + \vec{p} \times H^*_t &= c(\text{curl } H^* - \frac{4\pi\sigma}{c} E^*) \\ -\vec{p} \times E^*_t + \mu H^*_t &= -c \text{ curl } E^* \ . \end{aligned} \tag{5.7}$$

We symmetrize the equations (5.7) by adding an artificial "magnetic conductivity" k so that

$$\begin{aligned} \epsilon E^*_t + \vec{p} \times H^*_t &= c(\text{curl } H^* - \frac{4\pi\sigma}{c} E^*) \\ -\vec{p} \times E^*_t + \mu H^*_t &= -c(\text{curl } E^* + \frac{4\pi k}{c} H^*) \ . \end{aligned} \tag{5.8}$$

We shall of course make k = 0 afterwards. But the symmetry of the above equations allows us to derive from any relation a second one simply by replacing E^*, H^*, ϵ, μ, σ, k by H^*, E^*, $-\mu$, $-\epsilon$, $-k$, $-\sigma$, respectively. We multiply the first equation by μ and subtract the vector product of $\vec{p}$ with the second equation. It follows since $\vec{p}^{\,2} = \epsilon\mu$ that

$$\frac{1}{c}\,(\vec{p} \cdot E^*_t)\,\vec{p} = \mu \text{ curl } H^* + \vec{p} \times \text{curl } E^* + \frac{4\pi}{c}\left[k(\vec{p} \times H^*) - \sigma\mu E^*\right],$$

or since by (4.7) and (4.12) $H^* = \frac{1}{\mu}(\vec{p} \times E^*)$ and $\vec{p} \times H^* = -\epsilon E^*$, that the differential expression obtained by substitutions on the right side, namely,

$$\text{curl } (\frac{\vec{p}}{\mu} \times E^*) + \frac{\vec{p}}{\mu} \times \text{curl } E^* - \frac{4\pi}{c\mu}(k\epsilon + \sigma\mu)E^* \tag{5.9}$$

must yield a vector which has the direction of $\vec{p}$.

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[7] In the equations (5.6) E^*_t means partial differential of E(x,y,s,t) with respect to t and then substitution of $\frac{\psi}{c}$ for t. Curl E on the right side means curl E(x,y,z,t) and then substitute $\frac{\psi}{c}$ for t. Likewise for curl H.

Consider now the vector $E^* \times \operatorname{curl} \frac{\vec{p}}{\mu}$. Since $\operatorname{curl} \frac{\vec{p}}{\mu} = \frac{1}{\mu} \operatorname{curl} \vec{p} + \vec{p} \times \operatorname{grad} \frac{1}{\mu}$ and since $\operatorname{curl} \vec{p} = 0$ by virtue of $\vec{p} = \operatorname{grad} \psi$ we conclude that

$$E^* \times \operatorname{curl} \frac{\vec{p}}{\mu} = E^* \times (\vec{p} \times \operatorname{grad} \frac{1}{\mu}) \ .$$

In view of (4.13), the right side equals $(E^* \cdot \operatorname{grad} \frac{1}{\mu}) \vec{p}$ and so the left side also has the direction of $\vec{p}$. Consequently this expression may be added to (5.9) and we find that

$$(5.10) \quad \operatorname{curl} (\frac{\vec{p}}{\mu} \times E^*) + \frac{\vec{p}}{\mu} \times \operatorname{curl} E^* + E^* \times \operatorname{curl} \frac{\vec{p}}{\mu} - \frac{4\pi}{c\mu} (k\epsilon + \sigma\mu) E^*$$

has the direction of $\vec{p}$.

We now make use of the vector identity

$$(5.11) \quad \operatorname{curl} (A \times E) + A \times \operatorname{curl} E + E \times \operatorname{curl} A$$

$$= -2(A_1 \frac{\partial E}{\partial x} + A_2 \frac{\partial E}{\partial y} + A_3 \frac{\partial E}{\partial z}) - E \operatorname{div} A + A \operatorname{div} E + \operatorname{grad}(A \cdot E)$$

which holds for any two vectors A and E. We introduce $A = \frac{\vec{p}}{\mu}$ and conclude, since $\vec{p} \cdot E^* = 0$, that

$$(5.12) \quad 2(p \frac{\partial E^*}{\partial x} + q \frac{\partial E^*}{\partial y} + r \frac{\partial E^*}{\partial z}) + E^* \mu \operatorname{div} \frac{\vec{p}}{\mu} + \frac{4\pi}{c} (k\epsilon + \sigma\mu) E^*$$

must have the direction of $\vec{p}$.

Let us now assume that the rays associated with a set of wavefronts are given in terms of a parameter τ so that λ in (5.4) is 1. Then $x(\tau)$, $y(\tau)$, $z(\tau)$ satisfy the equation

$$(5.13) \quad \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = n^2$$

and hence $\dot{x} = p$, $\dot{y} = q$, $\dot{z} = r$. With this assumption we may write (5.12) as follows, leaving out the asterisk for the moment,

$$(5.14) \quad \frac{dE}{d\tau} + \frac{1}{2} E \mu \operatorname{div} \frac{\vec{p}}{\mu} + \frac{2\pi}{c} (k\epsilon + \sigma\mu) E = \lambda \vec{p}$$

where the factor λ is as yet definite but undetermined.

By taking the scalar product of $\vec{p}$ with the equation (5.14), since $\vec{p} \cdot E^* = 0$, we obtain

$$\lambda = \frac{1}{n^2} (\vec{p} \cdot \frac{dE}{d\tau}) = - \frac{1}{n^2} (E \cdot \frac{d\vec{p}}{d\tau})$$

But since $p = \dot{x}$, $q = \dot{y}$ and $r = \dot{z}$, and since $|\vec{p}| = n$,

$$\frac{d\vec{p}}{d\tau} = p\frac{\partial \vec{p}}{\partial x} + q\frac{\partial \vec{p}}{\partial y} + r\frac{\partial \vec{p}}{\partial z} = \frac{1}{2}\,\text{grad}\,n^2 = n\,\text{grad}\,n\,.$$

Hence

$$\lambda = -E\cdot\frac{\text{grad}\,n}{n}\,. \tag{5.15}$$

We finally introduce the notation

$$= \mu\left(\frac{\psi_x}{\mu}\right)_x + \mu\left(\frac{\psi_y}{\mu}\right)_y + \mu\left(\frac{\psi_z}{\mu}\right)_z = \mu\,\text{div}\,\frac{\vec{p}}{\mu}\,.$$

The operator $\Delta_\mu\psi$ becomes Laplace's operator $\Delta\psi$ if $\mu = 1$.

Substitution in (5.14) yields

$$\frac{dE}{d\tau} + \frac{1}{2}\Delta_\mu\psi\,E + \left(E\cdot\frac{\text{grad}\,n}{n}\right)\text{grad}\,\psi + \frac{2\pi}{c}(k\epsilon + \sigma\mu)\,E = 0. \tag{5.16}$$

A similar equation will be found for H by replacing ϵ, μ, σ, k by $-\mu, -\epsilon, -k, -\sigma$. We then let $k = 0$ and have the two vector equations (restoring the asterisk)

$$\begin{aligned} &\frac{dE^*}{d\tau} + \frac{1}{2}\left(\Delta_\mu\psi + \frac{4\pi}{c}\sigma\mu\right)E^* + \left(E^*\cdot\frac{\text{grad}\,n}{n}\right)\text{grad}\,\psi = 0 \\ &\frac{dH^*}{d\tau} + \frac{1}{2}\left(\Delta_\epsilon\psi + \frac{4\pi}{c}\sigma\mu\right)H^* + \left(H^*\cdot\frac{\text{grad}\,n}{n}\right)\text{grad}\,\psi = 0 \end{aligned} \tag{5.17}$$

wherein

$$\Delta_\epsilon\psi = \epsilon\left[\left(\frac{\psi_x}{\epsilon}\right)_x + \left(\frac{\psi_y}{\epsilon}\right)_y + \left(\frac{\psi_z}{\epsilon}\right)_z\right]$$

$$\Delta_\mu\psi = \mu\left[\left(\frac{\psi_x}{\mu}\right)_x + \left(\frac{\psi_y}{\mu}\right)_y + \left(\frac{\psi_z}{\mu}\right)_z\right]$$

These equations are differential equations along a given ray $x(\tau), y(\tau), z(\tau)$. They allow us to find E^* and H^* if these vectors are known at any particular point of the ray. We have but to substitute the quantities ϵ, μ, σ, n, $\Delta\psi$, and grad ψ, wherein x, y, and z have the values $x(\tau)$, $y(\tau)$ and $z(\tau)$ for the ray, and then solve the resulting first order differential equations for E^* and H^*.

5.4. We can simplify the equations (5.17) considerably. It follows from the definition of S, and equations (4.19) and (5.1) that, in an isotropic medium, $\vec{p} = \epsilon\mu\vec{s}^*$. Consequently

$$\Delta_\mu\psi = \mu\,\text{div}\,\frac{\vec{p}}{\mu} = \mu\,\text{div}\,\epsilon\,\vec{s}^* = n^2\,\text{div}\,\vec{s}^* + s^*\cdot\text{grad}\,\epsilon\,.$$

This gives, through $\mu \vec{s}^* = \frac{1}{\epsilon}\vec{p}$ and $\vec{p} \cdot \text{grad}\,\epsilon = \frac{d\epsilon}{d\tau}$ (since the rays have the direction of $\vec{p}$) that

$$\Delta_{\mu}\psi = n^2 \operatorname{div} \vec{s}^* + \frac{d}{d\tau} \log \epsilon$$

and similarly

$$\Delta_{\epsilon}\psi = n^2 \operatorname{div} \vec{s}^* + \frac{d}{d\tau} \log \mu \, .$$

With the aid of these expressions we write (5.17) as follows:

$$\frac{d(\sqrt{\epsilon}\, E^*)}{d\tau} + \frac{1}{2}\left(n^2 \operatorname{div} \vec{s}^* + \frac{4\pi}{c}\sigma\mu\right)(\sqrt{\epsilon}\, E^*) + \left(\sqrt{\epsilon}\, E^* \cdot \frac{\text{grad}\, n}{n}\right)\text{grad}\,\psi = 0$$

(5.18)

$$\frac{d(\sqrt{\mu}\, H^*)}{d\tau} + \frac{1}{2}\left(n^2 \operatorname{div} \vec{s}^* + \frac{4\pi}{c}\sigma\mu\right)(\sqrt{\mu}\, H^*) + \left(\sqrt{\mu}\, H^* \cdot \frac{\text{grad}\, n}{n}\right)\text{grad}\,\psi = 0,$$

so that $\sqrt{\epsilon}\, E^*$ and $\sqrt{\mu}\, H^*$ satisfy the same differential equation.

We now introduce two vectors P and Q by the equations

(5.19)

$$\sqrt{\epsilon}\, E^* = \sqrt{4\pi W}\; P$$

$$\sqrt{\mu}\, H^* = \sqrt{4\pi W}\; Q$$

where W is, for the moment, an unknown function. These vectors have the same direction as E* and H* but different length. We expect them to be unit vectors.

We introduce (5.19) in (5.18) and get

$$\frac{1}{2} P\left(\frac{d \log W}{d\tau} + n^2 \operatorname{div} \vec{s}^* + \frac{4\pi}{c}\sigma\mu\right) + \frac{dP}{d\tau} + \left(\frac{P \cdot \text{grad}\, n}{n}\right)\text{grad}\,\psi = 0$$

and an identical equation for Q. We now submit W to the condition

(5.20)
$$\frac{d}{d\tau} \log W = -\left(n^2 \operatorname{div} \vec{s}^* + \frac{4\pi}{c}\sigma\mu\right)$$

and thus obtain for P and Q the simple equations

(5.21)
$$\frac{dP}{d\tau} + \frac{P \cdot \text{grad}\, n}{n}\, \text{grad}\,\psi = 0$$

$$\frac{dQ}{d\tau} + \frac{Q \cdot \text{grad}\, n}{n}\, \text{grad}\,\psi = 0 \quad .$$

We shall integrate equation (5.20) after establishing a helpful subsidiary relationship. Consider a "tube" of rays, which as indicated in Fig. 7, is bounded by two sections w' and w of wavefronts. To each surface element dw of w there belongs a corresponding surface element dw' of w', namely the one which is cut out by the rays from dw. Let

(5.22)
$$dw' = \frac{K}{K'}\, dw$$

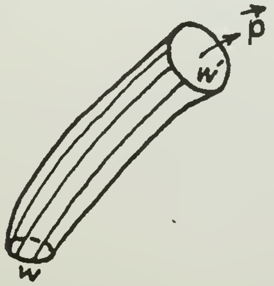

Fig. 7

where $K(\tau)$ is a function which measures the expansion of the wave fronts along a ray. Since by (5.13) the geometrical line element $d\sigma$ on a ray is given by $d\sigma = n d\tau$ we can write the volume element dxdydz in the form

(5.23) $$dxdydx = dw\, d\sigma = ndwd\tau \quad .$$

Now let us apply Gauss' theorem to our tube of rays thus:

$$\iiint \mathrm{div}\, \vec{s}^{\,*}\, dxdydz = \iint_{w'} \vec{s}^{\,*} \cdot \vec{n}' dw' - \iint_{w} \vec{s}^{\,*} \cdot \vec{n}\, dw = \iint_{w'} |\vec{s}^{\,*}|\, dw' - \iint_{w} |\vec{s}^{\,*}|\, dw \ ,$$

where $\vec{n}$ and $\vec{n}'$ are unit normals to dw and dw' respectively. Since $\vec{p} = n^2 \vec{s}^{\,*}$ and $|\vec{p}| = n$, $|\vec{s}^{\,*}| = \frac{1}{n}$. Hence, with (5.22), the last expression equals $\iint_w \left(\frac{1}{n' K(\tau_1)} - \frac{1}{nK(\tau_0)} \right) K(\tau_0)\, dw$ (where τ_0 is the value at w and τ_1 the value at w')

$$= \iint_w \left[\int_{\tau_0}^{\tau_1} \frac{d}{d\tau} \left(\frac{1}{nk(\tau)} \right) d\tau \right] k(\tau_0)\, dw = \iint_w \left[\int_{\tau_0}^{\tau_1} \frac{d}{d\tau} \left(\frac{1}{nk(\tau)} \right) nk(\tau_0)\, \frac{d\tau}{n} \right] dw$$

$$= \iiint \frac{d}{d\tau} \left(\frac{1}{nk(\tau)} \right) k(\tau_0) n\, \frac{ndwd\tau}{n^2} \ .$$

Hence it follows from (5.23) that at $\vec{s}^{\,*}(\tau_0)$

(5.24) $$\mathrm{div}\, \vec{s}^{\,*} = \left[-\frac{d}{d\tau} (\log(nK)) \right]_{\tau=\tau_0} \frac{1}{n^2} \quad .$$

With this result (5.24) we may write (5.20) as

(5.25) $$\frac{W}{nK} = \frac{W_0}{n_0 K_0}\, e^{-\frac{4\pi}{c} \int_{\tau_0}^{\tau} \sigma\mu \, d\tau}$$

The factor W formally introduced in (5.19) is the electromagnetic energy, for from (5.19) $\frac{E^* \times H^*}{4\pi W} = \frac{P \times Q}{\sqrt{\epsilon\mu}}$. Now P and Q can be made to be unit vectors by choice of the constant of integration in (5.20). Since $E^* \times H^*$ has the direction of $\vec{p}$, we have that $\frac{E^* \times H^*}{4\pi W} = \frac{\vec{p}}{|\vec{p}|\sqrt{\epsilon\mu}} = \frac{\vec{p}}{\epsilon\mu} = \vec{s}^{\,*}$. Reference to W in article 2.3 shows that the W introduced here is electromagnetic energy if the constant of integration in (5.20) is properly chosen.

Several other implications of equations (5.21) will be noted. We form the scalar product of gradψ with the first of equations (5.21). It follows that

$$\operatorname{grad}\psi \cdot \frac{dP}{d\tau} + n(P \cdot \operatorname{grad} n) = 0.$$

However, $n \operatorname{grad} n = \frac{d}{d\tau} \operatorname{grad}\psi$; hence

$$\frac{d}{d\tau}(P \cdot \operatorname{grad}\psi) = 0. \tag{5.26}$$

Thus, if $(P \cdot \operatorname{grad}\psi) = 0$ at one particular point of the ray, then it will be zero at any point of this ray. The same is true for the product $(Q \cdot \operatorname{grad}\psi)$. We next multiply the first of equations (5.21) by P and use that $P \cdot \operatorname{grad}\psi = 0$. It follows that $P \cdot \frac{dP}{d\tau} = 0$, i.e., $P^2 = \text{const.}$ along the ray. Thus, if P is a unit vector at one point of the ray it will be a unit vector at all points. Finally we get from (5.21) that

$$Q \cdot \frac{dP}{d\tau} + P \frac{dQ}{d\tau} = \frac{d}{d\tau}(P \cdot Q) = 0,$$

i.e., the vectors P and Q are normal to each other if they are normal at one particular point of the ray.

From (5.25) it follows that W = 0 on all points of a ray if $W_0 = 0$ at a particular point. Also if the medium is non-absorbing then $\sigma = 0$ and $\frac{W}{nK}$ is constant along a ray.

Finally, $W(\tau)$ is completely determined by its value $W(\tau_0)$ at a particular point along a ray. We can thus construct the vectors E* and H* on any wavefront $\psi = ct$ if they are given on one particular wave-front $\psi = ct_0$ . .

This result means that if we have a signal such that discontinuities of the electromagnetic field are different from 0 only on a section Γ of a wave-front then the rays through Γ determine the region where the signal will be observed. This does not exclude the possibility that the signal will penetrate into the region of geometrical shadow. However this diffracted excitation does not have a sudden discontinuous beginning, that is, it is not given by geometrical optics, with which we are now dealing.

6. The Behavior of Wave-Fronts at Refracting Surfaces.

6.1. The theory of wave-fronts presented thus far has assumed that ϵ, μ, and σ are continuous and have continuous derivatives and the wave-fronts have therefore progressed continuously through space. This article will consider the

effect of discontinuities in ϵ, μ, and σ on the behavior of the wave-fronts. Physically the presence of obstacles or of lenses in a medium introduces such discontinuities. The surfaces on which the discontinuities in ϵ, μ, and σ occur are not limited to planes. Nevertheless it is shown that the usual laws of reflection and refraction are obeyed by the wave-fronts. Moreover, the reflection and transmission coefficients are obtained, and these prove to be identical in form with the Fresnel formulas for reflection and transmission of a plane wave at a plane surface.

6.2. The effect of discontinuities in ϵ, μ, and σ is to split up the wave-front into several parts. In addition to a transmitted wave-front there is a reflected one. If several refracting surfaces are present then a rather complicated system of multiple wave-fronts which overlap each other is established in time. This overlapping is avoided if the wave-fronts $\psi = ct$ are represented in the four dimensional x,y,z,t space by a hyper-surface $\phi = \psi - ct = 0$. It consists of a number of branches as is schematically illustrated in Fig. 8 for the x,y,t space. The branches are joined together in groups of three on the refracting surfaces, however at different "heights", i.e., at different times.

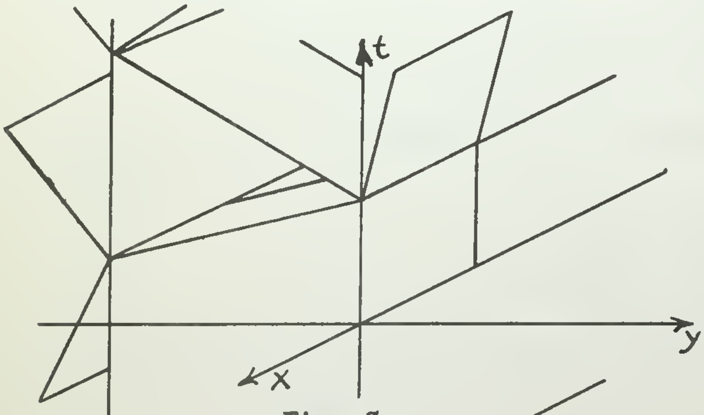

Fig. 8

Consider at such a triple of branches joined together on the refracting surface Γ :

$$x = f(\xi, \eta)$$
$$(6.1) \qquad y = g(\xi, \eta)$$
$$z = h(\xi, \eta)$$

The three hyper surfaces given by:

$$(6.2) \quad \psi - ct = 0, \; \psi^* - ct = 0, \text{ and } \psi' - ct = 0$$

represent the incident, reflected, and refracted wave-fronts, respectively.

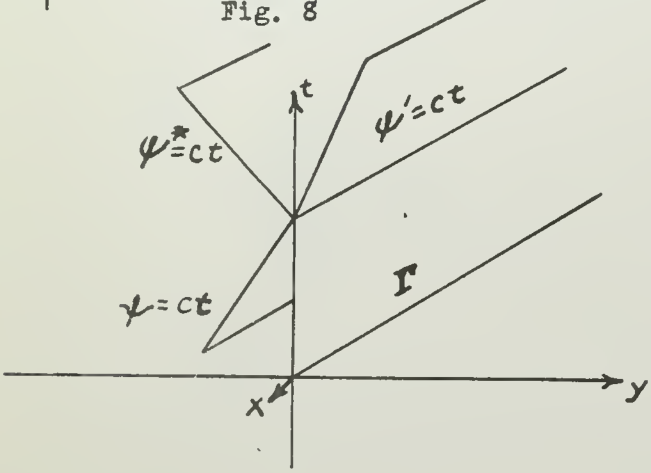

Fig. 9

In order that these hyper surfaces be joined together on Γ we must require that $\psi = \psi^* = \psi'$ on Γ. This means that

$$(6.3)\quad \psi(f(\xi,\eta)g,(\xi,\eta),\ h(\xi,\eta)) = \psi^*(f(\xi,\eta),g(\xi,\eta),h(\xi,\eta) = \psi'(f(\xi,\eta),g(\xi,\eta),h(\xi,\eta))$$

If we differentiate these equations with respect to ξ or η we obtain interior derivatives of ψ, ψ^*, ψ' which again must be equal. It follows that

$$(6.4)\quad \begin{aligned} (\psi^*_x - \psi_x)f_\xi + (\psi^*_y - \psi_y)g_\xi + (\psi^*_z - \psi_z)h_\xi &= 0 \\ (\psi^*_x - \psi_x)f_\eta + (\psi^*_y - \psi_y)g_\eta + (\psi^*_z - \psi_z)h_\eta &= 0 \end{aligned}$$

and

$$\begin{aligned} (\psi'_x - \psi_x)f_\xi + (\psi'_y - \psi_y)g_\xi + (\psi'_z - \psi_z)h_\xi &= 0 \\ (\psi'_x - \psi_x)f_\eta + (\psi'_y - \psi_y)g_\eta + (\psi'_z - \psi_z)h_\eta &= 0 \end{aligned}$$

These equations show that the differences $\vec{p}^* - \vec{p}$ and $\vec{p}' - \vec{p}$ of the wave normals

$$\vec{p} = (\psi_x, \psi_y, \psi_z);\ \vec{p}^* = (\psi^*_x, \psi^*_y, \psi^*_z)\ ,\ \vec{p}' = (\psi'_x, \psi'_y, \psi'_z)$$

are parallel to the normal $\vec{M}$ of the refracting surface at the point ξ, η. We write

$$(6.5)\quad \begin{aligned} \vec{p}^* &= \vec{p} + \Gamma_1 \vec{M} \\ \vec{p}' &= \vec{p} + \Gamma_2 \vec{M} \end{aligned}$$

and conclude that all four vectors $\vec{M}, \vec{p}, \vec{p}^*, \vec{p}'$ lie in one and the same plane, the plane of incidence. We also conclude from (6.5) that

$$(6.6)\quad \begin{aligned} \vec{p}^* \times \vec{M} &= \vec{p} \times \vec{M} \\ \vec{p}' \times \vec{M} &= \vec{p} \times \vec{M} \end{aligned}$$

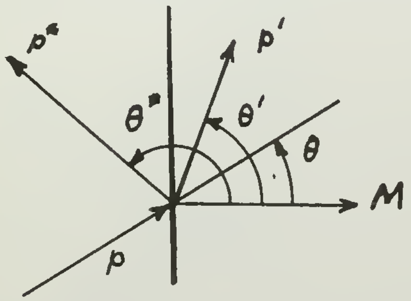

Fig. 10

By taking the absolute value of these equations and remembering that $|\vec{p}|^2 = |\vec{p}^*|^2 = n^2;\ |\vec{p}'|^2 = n'^2$ we get the law of reflection and Snell's law of refraction:

$$(6.7)\quad \begin{aligned} \sin\ \theta^* &= \sin\theta \\ n'\sin\ \theta' &= n\sin\theta \end{aligned}$$

The physically consistent solutions of these equations are

$$\theta^* = \pi - \theta$$

and an angle θ' of the interval $0 \leqq \theta' \leqq \pi/2$ provided that $\vec{M}$ is oriented relative to $\vec{p}$ such that $0 \leqq \theta \leqq \pi/2$. The eouation $n' \sin \theta' = n \sin \theta$ has no solution if $\frac{n}{n'} \sin \theta > 1$. This is the case of total reflection where the signal is not transmitted through the refracting surface.

6.3. We study next the electromagnetic field in the neighborhood of the manifold J where the three hypersurfaces $\psi = ct = 0$, $\psi^* - ct = 0$ and $\psi' - ct = 0$ intersect each other. This manifold J lies on the cylindrical hypersurface $\Gamma(x,y,z) = 0$ which has the refracting surface as basis. The neighborhood of J is divided into five regions by the four hyper-surfaces. Let us assume that the vectors E and H[8] are continuous in these regions and that they attain finite limits

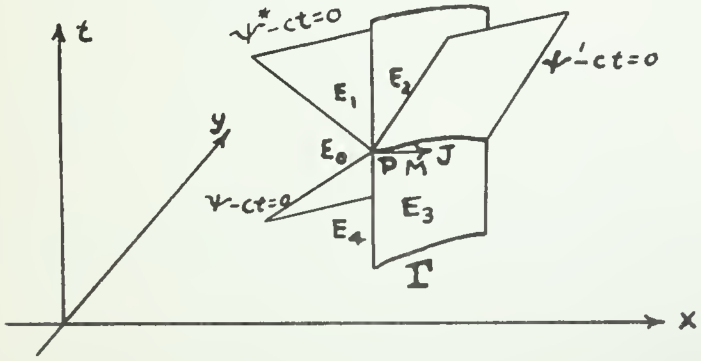

Fig. 11

$$E_0, E_1, E_2, E_3, E_4$$
$$H_0, H_1, H_2, H_3, H_4$$

if point P of the manifold J is approached.

We denote the surface normal of the refracting surface by M and consider the limits E_1, E_2, E_3, E_4 and H_1, H_2, H_3, H_4. By applying the conditions (3.12) derived in article 3.5 we find that

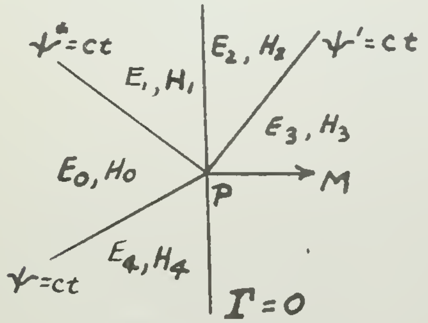

Fig. 12

$$(E_1 - E_2) \times M = 0 \quad (H_1 - H_2) \times M = 0$$

(6.8)

$$(E_4 - E_3) \times M = 0 \quad (H_4 - H_3) \times M = 0$$

---

[8] We shall denote the incident wave-front by E and H instead of E* and H* and shall use E* and H* for the reflected wave-front in this article.

We next consider the boundary values of the discontinuities on the hypersurfaces $\psi = ct$, $\psi^* = ct$, $\psi' = ct$. They are given by

$$(6.9)\qquad \begin{aligned} E &= E_0 - E_4 & H &= H_0 - H_4 \\ E^* &= E_1 - E_0 & H^* &= H_1 - H_0 \\ E' &= E_2 - E_3 & H' &= H_2 - H_3 \end{aligned}$$

and represent the incident, reflected and transmitted signal at P respectively. It follows readily by (6.8) that

$$(6.10)\qquad \begin{aligned} (E + E^*) \times M &= E' \times M \\ (H + H^*) \times M &= H' \times M \end{aligned}$$

These discontinuities satisfy also the vector equations

$$(6.11)\qquad \begin{aligned} &\vec{p} \times H + \epsilon E = 0, \quad \vec{p}^* \times H^* + \epsilon E^* = 0, \quad \vec{p}' \times H' + \epsilon' E' = 0 \\ &\vec{p} \times E - \mu H = 0, \quad \vec{p}^* \times H^* - \mu H^* = 0, \quad \vec{p}' \times E' - \mu' H' = 0 \end{aligned}$$

where $\vec{p} = \operatorname{grad} \psi$, $\vec{p}^* = \operatorname{grad} \psi^*$, $\vec{p}' = \operatorname{grad} \psi'$ are the wave normals of the three wave-fronts.

6.4. We have seen that these three wave normals and the vector M lie in one and the same plane normal to a unit vector S defined by

$$(6.12)\qquad \vec{p} \times M = \vec{p}^* \times M = \vec{p}' \times M = kS$$

The constant k equals

$$k = n \sin \theta = n \sin \theta^* = n' \sin \theta'$$

In addition to S we introduce three other unit vectors

$$(6.13)\qquad N = S \times \frac{\vec{p}}{n}, \quad N^* = S \times \frac{\vec{p}^*}{n}, \quad N' = S \times \frac{\vec{p}'}{n'}$$

which are normal to the incoming, reflected and refracted ray respectively. The vectors N,S; N*, S; N',S thus form three pairs of orthogonal unit vectors at right angles attached to each of the three rays. Consequently we may express the vectors E,H; E*,H*; E',H' as linear combinations of the associated vectors N and S. We verify readily that the equations (6.11) are satisfied if we let

$$(6.14)\qquad \begin{aligned} &\sqrt{\epsilon}\, E = \alpha N + \beta S\ ; \quad \sqrt{\epsilon}\, E^* = \alpha^* N^* + \beta^* S\ ; \quad \sqrt{\epsilon'}\, E' = \alpha' N' + \beta' S \\ &\sqrt{\mu}\, H = -\beta N + \alpha S\ ; \quad \sqrt{\mu}\, H^* = -\beta^* N^* + \alpha^* S; \quad \sqrt{\mu'}\, H' = -\beta' N' + \alpha' S \end{aligned}$$

We shall shortly introduce these expressions into (6.10). We have first from (6.14) that

$$\sqrt{\epsilon}\,(E \times M) = \alpha (N \times M) + \beta (S \times M)$$

Since by relations (6.12)

$$N \times M = (S \times \frac{p}{n}) \times M = -\cos\theta\, S$$

we get

$$(E \times M) = \frac{-\alpha}{\sqrt{\epsilon}} \cos\theta\, S + \frac{\beta}{\sqrt{\epsilon}} (S \times M)$$

and similarly

$$(H \times M) = \frac{\beta}{\sqrt{\mu}} \cos\theta\, S + \frac{\alpha}{\sqrt{\mu}} (S \times M) \tag{6.15}$$

Corresponding equations are found with regard to the vectors E*,H* and E',H'. Thus equations (6.10) yield, on account of $\cos\theta^* = -\cos\theta$, the four conditions

$$\begin{aligned} \alpha^* - \alpha &= -\sqrt{\frac{\epsilon}{\epsilon'}}\,\frac{\cos\theta'}{\cos\theta}\,\alpha' & \qquad \alpha^* + \alpha &= \sqrt{\frac{\mu}{\mu'}}\,\alpha' \\ \beta^* - \beta &= -\sqrt{\frac{\mu}{\mu'}}\,\frac{\cos\theta'}{\cos\theta}\,\beta' & \qquad \beta^* + \beta &= \sqrt{\frac{\epsilon}{\epsilon'}}\,\beta' \end{aligned} \tag{6.16}$$

which can be used to express α^*, β^*; α', β' in terms of the incident components α and β. The result is

$$\begin{aligned} \frac{\alpha^*}{\alpha} &= \frac{\sqrt{\frac{\mu}{\mu'}} - \sqrt{\frac{\epsilon}{\epsilon'}}\,\frac{\cos\theta'}{\cos\theta}}{\sqrt{\frac{\mu}{\mu'}} + \sqrt{\frac{\epsilon}{\epsilon'}}\,\frac{\cos\theta'}{\cos\theta}}\,, & \frac{\beta^*}{\beta} &+ \frac{\sqrt{\frac{\epsilon}{\epsilon'}} - \sqrt{\frac{\mu}{\mu'}}\,\frac{\cos\theta'}{\cos\theta}}{\sqrt{\frac{\epsilon}{\epsilon'}} + \sqrt{\frac{\mu}{\mu'}}\,\frac{\cos\theta'}{\cos\theta}} \\ \frac{\alpha'}{\alpha} &= \frac{2}{\sqrt{\frac{\mu}{\mu'}} + \sqrt{\frac{\epsilon}{\epsilon'}}\,\frac{\cos\theta'}{\cos\theta}}\,, & \frac{\beta'}{\beta} &= \frac{2}{\sqrt{\frac{\epsilon}{\epsilon'}} + \sqrt{\frac{\mu}{\mu'}}\,\frac{\cos\theta'}{\cos\theta}} \end{aligned} \tag{6.17}$$

In relation to the plane of incidence it is customary to call the components of the electric discontinuity E with regard to N and S the parallel and normal component of E. One uses the notation

$$\begin{aligned} A_p &= \frac{\alpha}{\sqrt{\epsilon}} & \qquad A_s &= \frac{\beta}{\sqrt{\epsilon}} \\ R_p &= \frac{\alpha^*}{\sqrt{\epsilon}} & \qquad R_s &= \frac{\beta^*}{\sqrt{\epsilon}} \\ D_p &= \frac{\alpha'}{\sqrt{\epsilon'}} & \qquad D_s &= \frac{\beta'}{\sqrt{\epsilon'}} \end{aligned} \tag{6.18}$$

It follows letting $\mu=\mu=1$; $\sqrt{\epsilon} = n$, $\sqrt{\epsilon'} = n'$ that

$$R_p = A_p \frac{1 - \frac{n}{n'}\frac{\cos\theta'}{\cos\theta}}{1 + \frac{n}{n'}\frac{\cos\theta'}{\cos\theta}} \quad ; \quad R_s = A_s \frac{\frac{n}{n'} - \frac{\cos\theta'}{\cos\theta}}{\frac{n}{n'} + \frac{\cos\theta'}{\cos\theta}}$$

$$(6.19)$$

$$D_p = A_p \frac{2\frac{n}{n'}}{1 + \frac{n}{n'}\frac{\cos\theta'}{\cos\theta}} \quad ; \quad D_s = A_s \frac{2\frac{n}{n'}}{\frac{n}{n'} + \frac{\cos\theta'}{\cos\theta}}$$

The above formulae are identical with Fresnel's formulae for the reflection and transmission of a plane wave on a plane surface or refraction. Their significance, however, is more general since they also give the reflection and transmission of any discontinuous signal along any ray which strikes a surface of refraction of arbitrary shape. We have to consider the equations (6.17) as the supplement of the transport equations for signals in a continuous medium.

We remark explicitly that these formulae are valid also if two absorbing media are separated by a surface of refraction. The conductivity, σ, thus is found to be without effect upon the reflection and transmission on the surface itself. It plays its part, however, in damping the signal when it proceeds from the point of incidence into the interior. We shall verify this result later by exact integration of Maxwell's equations. Though σ has a considerable influence upon the steady state solution it is without direct effect on the transient phenomena on a proceeding wave-front.

7. The Asymptotic Development of Steady State Dipole Fields.

7.1. The preceding articles dealt with the first excitation which reaches a given point in space from an electromagnetic source. This first excitation or wave-front follows certain laws which are quite independent of the field established afterwards. The relation of this theory to the problem of the paper will soon be apparent.

We turn now to the steady state fields set up by the source, and we shall investigate the problem set for this paper, namely, to obtain asymptotic expressions for the steady state fields in terms of the variable $k = \frac{2\pi}{\lambda}$.

Before developing the general theory we shall illustrate the nature of it by treating the field of a dipole in a homogeneous as well as in a non-homogeneous medium. The mathematics of the dipole field used here is not new. However, we shall regard it in the light of our general theory and shall gain insight into the significance of the terms.

It will be apparent that for small wave lengths the amplitudes of the field vectors of the steady state field satisfy the same conditions as do the electromagnetic vectors on a wave-front.

7.2. To treat steady-state fields we return to Maxwell's equations as presented in article 2 and to the various formulations of these equations given in that article.

We shall assume first that we are dealing with a homogeneous medium in which, for simplicity, $\epsilon = \mu = 1$ and $\sigma = 0$.

Enforced periodic currents shall be given by the vector field

$$\frac{1}{4\pi} R(F_t) = R\left\{- i\omega \vec{g}\, e^{-i\omega t}\right\}, \tag{7.1}$$

where $F = \vec{g}(x,y,z)e^{-i\omega t}$. The steady state field which is established under the influence of this current distribution must satisfy equation (2.8) which in our special case reduces to

$$\begin{aligned} \operatorname{curl} v + iku &= -ik\vec{g} \\ \operatorname{curl} u - ikv &= 0. \end{aligned} \tag{7.2}$$

We solve these equations as follows. We define first a vector field A(x,y,z) by the integral

$$\vec{A} = -\frac{ik}{4\pi}\iiint \vec{g}(\xi, \eta, \zeta)\frac{e^{ikr}}{r}\, d\xi\, d\eta\, d\zeta \tag{7.3}$$

where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$

Let us assume that $\vec{g}(\xi, \eta, \zeta)$ is sectionally smooth i.e. that any finite domain may be divided into finitely many subdomains in which $\vec{g}$ is continuous and has continuous derivatives. Then in any of these subdomains $\vec{A}$ satisfies the differential equation in x,y, and z,

$$\Delta\vec{A} + k^2\vec{A} = ik\vec{g}. \tag{7.4}$$

This statement may be verified by the use of Green's theorem.

We now construct two vector fields u and v with the aid of the vector field $\vec{A}$, namely,

$$(7.5)\qquad \begin{aligned} u &= ik\vec{A} - \frac{1}{ik}\ \text{grad div}\ \vec{A} \\ v &= \text{curl}\ \vec{A}, \end{aligned}$$

and demonstrate that u and v satisfy the equations (7.2). Indeed

$$\text{curl}\ u = ik\ \text{curl}\ \vec{A} = ikv,$$

i.e., the second equation (7.2) is satisfied. Furthermore

$$\text{curl}\ v + iku = \text{curl curl}\ \vec{A} - k^2 A - \text{grad div}\ \vec{A}$$

$$= -\Delta\vec{A} - k^2\vec{A} = -ik\vec{g},$$

i.e., the first equation (7.2) is satisfied.

We thus recognize in (7.5) solutions of the steady state equations (7.2). Of course, we cannot be sure that this particular solution introduced somewhat arbitrarily, represents the periodic field which will be established with increasing time under the influence of the enforced currents (7.1). All we have shown is that the integral satisfies Maxwell's equations (7.2). Actually however, this is the case as we shall verify in a following chapter.

7.3. Let us now assume that enforced currents are confined to a small neighborhood of the point $\xi = \eta = \zeta = 0$. Let

$$\vec{g} = \frac{3}{(ik)^2\delta^3}\vec{M} \qquad \text{for } \xi^2 + \eta^2 + \zeta^2 < \delta^2$$

$$= 0 \qquad \text{for } \xi^2 + \eta^2 + \zeta^2 > \delta^2$$

where $\vec{M}$ is an arbitrary constant vector and δ is arbitrary but small. We find for $\delta \rightarrow 0$ that

$$(7.7)\qquad \vec{A} = -\vec{M}\,\frac{e^{ikr}}{ikr}, \quad \text{where } r = \sqrt{x^2 + y^2 + z^2}$$

By inserting $\vec{A}$ into the general result (7.5) we get Hertz' solution for the radiation of a harmonic dipole of moment $\vec{M}$ in a vacuum, namely,

$$(7.8)\qquad \begin{aligned} v &= -(\text{grad}\ r \times \vec{M})\left(\frac{1}{r} - \frac{1}{ikr^2}\right)e^{ikr} \\ u &= -\frac{1}{ik}\ \text{curl}\ v. \end{aligned}$$

We note from (7.8) that the electromagnetic field of dipole source can be written in the following form

$$u = e^{ikr}\left(U_0 + \frac{1}{ik}\,U_1 + \frac{1}{(ik)^2}\,U_2\right)$$

(7.9)

$$v = e^{ikr}\left(V_0 + \frac{1}{ik}\,V_1\right)$$

where the vectors U_ν and V_ν are given by the relations

$$V_0 = -\frac{1}{r}(\operatorname{grad} r \times \vec{M}), \qquad U_0 = -\operatorname{grad} r \times V_0$$

(7.10)

$$V_1 = +\frac{1}{r^2}(\operatorname{grad} r \times \vec{M}), \qquad U_1 = -\operatorname{grad} r \times V_1 - \operatorname{curl} V_0$$

$$U_2 = -\operatorname{curl} V_1 \; .$$

Since these vectors are independent of k we conclude from (7.9) that with increasing values of $k = \frac{2\pi}{\lambda}$, i.e., for small wave length λ, the field is asymptotically given by the vector field

$$u = U_0\, e^{ikr}$$

(7.11)

$$v = V_0\, e^{ikr}$$

We also note, for a fixed value of k but large values of r, that the main contribution to the field comes from the expressions (7.11).

7.4. The vectors U_0 and V_0 which determine the asymptotic behavior of the field for large values of k or r are explicitly given by the expressions

$$U_0 = -\frac{1}{r}(\operatorname{grad} r \times \vec{M}) \times \operatorname{grad} r$$

(7.12)

$$V_0 = -\frac{1}{r}(\operatorname{grad} r \times \vec{M}) \; .$$

Because ϵ and μ are constant in our space the surfaces $\psi = r =$ const. represent a system of wave fronts. We may therefore interpret the unit vectors $\operatorname{grad} r = \vec{p}$ as wave normals and write (7.12) in the form

$$U_0 = -\frac{1}{r}(\vec{p} \times \vec{M}) \times \vec{p}$$

(7.13)

$$V_0 = -\frac{1}{r}(\vec{p} \times \vec{M}) \; .$$

We conclude that

$$U_0 \cdot \vec{p} = V_0 \cdot \vec{p} = U_0 \cdot V_0 = 0 \tag{7.14}$$

so that U_0 and V_0 are tangential to the wave fronts and normal to each other.

Furthermore from (7.13) it follows that

$$\begin{aligned} \vec{p} \times V_0 + U_0 &= 0 \\ \vec{p} \times U_0 - V_0 &= 0 \quad , \end{aligned} \tag{7.15}$$

and, since $\vec{p}$ = grad r is constant along a straight line through the origin, it follows from equations (7.12) that

$$\begin{aligned} \frac{d}{dr}(rU_0) &= 0 \\ \frac{d}{dr}(rV_0) &= 0. \end{aligned} \tag{7.16}$$

If we write these last equations in the form

$$\frac{dU_0}{dr} + \frac{1}{2}\Delta\psi\, U_0 = 0$$

$$\frac{dV_0}{dr} + \frac{1}{2}\Delta\psi\, V_0 = 0$$

where $\psi = r$ and $\Delta\psi$, accordingly, is $\frac{2}{r}$ we recognize by comparison with (5.17) that U_0 and V_0 satisfy the same relations as those derived for electromagnetic signals, i.e., for vectors E* and H* on the boundary of a region of penetration.

This significant fact, that the amplitudes of periodic waves of small wave length are propagated according to the same laws as electromagnetic signals, will be established shortly in general.

7.5. We consider the radiation problem for a time periodic current distribution (7.1) in the case of a non-homogeneous medium. The steady state solution u,v must satisfy the differential equations (2.8), namely,

$$\begin{aligned} \text{curl } v + ik\epsilon\, u - \frac{4\pi}{c}\sigma u &= -ik\vec{g} \\ \text{curl } u - ik\mu\, v &= 0. \end{aligned} \tag{7.18}$$

We assume that ϵ, μ, σ are continuous functions of x,y,z and that they have sectionally continuous derivatives.

We obtain the radiation from a dipole in a similar manner as before by choosing the special vector field given by (7.6) as our source. Since this source is zero except at $x = y = z = 0$, at all other points the field it gives rise to must satisfy the equations

$$\begin{aligned} &\operatorname{curl} v + ik\epsilon u - \frac{4\pi}{c}\sigma u = 0 \\ (7.19)\qquad &\\ &\operatorname{curl} u - ik\mu v = 0. \end{aligned}$$

In the neighborhood of $x = y = z = 0$, the radiation field will have the same singular behavior as the radiation solution for a homogeneous space in which $\epsilon = \epsilon_0$, $\mu = \mu_0$, $\sigma = \sigma_0$ are constant.

We consider next the wave fronts $\psi(x,y,z) = \text{const}$ of our medium around the point $x = y = z = 0$. We know that the wave function ψ satisfies the differential equation (4.15), namely,

$$(7.20)\qquad \psi_x^2 + \psi_y^2 + \psi_z^2 = \epsilon\mu = n^2 .$$

In view of the results for the homogeneous medium (7.9) in particular, we can make the attempt to represent the solution u,v of Maxwell's equations (7.19) by expressions of the form

$$\begin{aligned} &u = \sum_{\nu=0}^{\infty} \frac{1}{(ik)^\nu} \; U_\nu \; (x,y,z) \; e^{ik\psi} \\ (7.21)\qquad &\\ &v = \sum_{\nu=0}^{\infty} \frac{1}{(ik)^\nu} \; V_\nu \; (x,y,z) \; e^{ik\psi} \end{aligned}$$

Let us derive the necessary conditions for the coefficients U_ν, V_ν in such a development.

We introduce the infinite series (7.21) into the differential equations (7.19) and group terms in accordance with the power of k. We may then equate to zero the coefficient of each power of k and we find that the coefficients U_ν and V_ν must satisfy the conditions

$$\begin{aligned} &\operatorname{grad}\psi \times V_0 + \epsilon U_0 = 0 \\ (7.22)\qquad &\\ &\operatorname{grad}\psi \times U_0 - \mu V_0 = 0 \end{aligned}$$

and

$$\begin{aligned} &\operatorname{grad}\psi \times V_\nu + \epsilon U_\nu = -\left(\operatorname{curl} V_{\nu-1} - \frac{4\pi}{c}\sigma U_{\nu-1}\right) \\ (7.23)\qquad &\\ &\operatorname{grad}\psi \times U_\nu - \mu V_\nu = -\operatorname{curl} U_{\nu-1} \qquad \nu \geq 1. \end{aligned}$$

7.7. Consider first the quantities U_o and V_o which, in view of (7.21) determine the electromagnetic field for small wave length. We have, introducing the time variation,

$$E \approx R\left\{U_o e^{ik(\psi - ct)}\right\} \tag{7.24}$$
$$H \approx R\left\{V_o e^{ik(\psi - ct)}\right\}$$

and we recognize that wave fronts $\psi =$ const. represent surfaces at equal phase for these waves.

The equations (7.22) are formally identical with the conditions (4.12) for discontinuities on wave fronts. By substituting one equation of (7.22) into the other we see that the two equations can be satisfied by vectors $U_o \neq 0$ and $V_o \neq 0$ only if ψ is a solution of the equation $\psi_x^2 + \psi_y^2 + \psi_z^2 = n^2$. A development of the form (7.21) thus is possible only if the function ψ satisfies this differential equation. The difference between the equations (7.22) and the equations (4.12) for wave front discontinuities is that U_o and V_o are in general complex vectors.

We conclude readily from the equations (7.22) that

$$U_o \cdot \operatorname{grad}\psi = V_o \cdot \operatorname{grad}\psi = U_o \cdot V_o = 0 \tag{7.25}$$

i.e. U_o and V_o are tangential to the wave fronts and normal to each other.

We consider next the equations (7.23) in case $\nu = 1$. We have

$$\operatorname{grad}\psi \times V_1 + \epsilon U_1 = -\left(\operatorname{curl} V_o - \frac{4\pi}{c}\sigma U_o\right) \tag{7.26}$$
$$\operatorname{grad}\psi \times U_1 - \mu V_1 = -\operatorname{curl} U_o\ .$$

These equations are formally identical with the equations (5.7) for the vectors $-\frac{1}{c}E_t^*$, $-\frac{1}{c}H_t^*$, E^*, & H^* on a wave front. By the same argument we conclude that they are solvable only if the right sides satisfy certain conditions. The former method of deriving these conditions can be repeated literally. We obtain the result: the amplitude vectore U_o and V_o satisfy the following differential equations along the rays:

$$\frac{dU_o}{d\tau} + \frac{1}{2}\left(\Delta_\mu\psi + \frac{4\pi}{c}\sigma\mu\right)U_o + \left(U_o \cdot \frac{\operatorname{grad} n}{n}\right)\operatorname{grad}\psi = 0 \tag{7.27}$$
$$\frac{dV_o}{d\tau} + \frac{1}{2}\left(\Delta_\epsilon\psi + \frac{4\pi}{c}\sigma\mu\right)V_o + \left(V_o \cdot \frac{\operatorname{grad} n}{n}\right)\operatorname{grad}\psi = 0$$

which are identical with (5.17). Hence in the case of the dipole in the non-homogeneous medium it is again true that for small wave lengths the behavior of the steady state field is the behavior of signals on a wave front which in turn is the behavior of geometrical optics.

7.8. Our treatment of the steady state dipole field has thus far assumed that ϵ, μ, σ are either constant or continuous. The asymptotic representation assumed for the steady state field is relatively simple in these cases, the more complex being (7.21) for the continuous case. If ϵ, μ, and σ are discontinuous then the asymptotic expansion becomes more complex.

Discontinuities in ϵ, μ, and σ are generally caused by the presence of refracting surfaces. We therefore consider the case that our medium contains a number of refracting surfaces Γ on which the functions ϵ, μ, σ are discontinuous. We assume that a point source is located at the origin so that the vector field $\vec{g}(x,y,z)$ is zero except in the neighborhood of the point $x = y = z = 0$. Let us also assume that the refracting surfaces are either closed or extend towards infinity and that the space is divided into a finite number of domains in which ϵ, μ, σ are continuous. On account of multiple reflections we obtain, in each of these domains, infinitely many sets of wave fronts ψ_α = const. In view of the former results we attempt to represent the steady state solution u, v in form of an infinite series

$$(7.28)\qquad \begin{aligned} u &= \sum U_\alpha \, e^{ik\psi_\alpha} \\ v &= \sum V_\alpha \, e^{ik\psi_\alpha} \end{aligned}$$

where each individual term $U_\alpha \, e^{ik\psi_\alpha}$ and $V_\alpha \, e^{ik\psi_\alpha}$ satisfies Maxwell's equations. The representation (7.28) is a generalization of the representation (7.21) in that it allows for many sets of wave fronts possibly an infinite number, passing any given point (x,y,z) in space at which the steady state field u,v is being computed. Each U_α and V_α is assumed to be an infinite series of the form given on the right side of (7.21), that is, each U_α (and V_α) is of the form $\sum_{\nu=0}^{\infty} \frac{V_{\alpha,\nu}}{(ik)^\nu}$

To determine the various $U_{\alpha,\nu}$ and $V_{\alpha,\nu}$ it is necessary to use the usual electromagnetic boundary conditions at the surfaces Γ. It is not difficult to determine the field emanating directly from the source and transmitted through any one of the surfaces Γ. In principle it is also possible to construct the total

small wave length field, i.e., to find the various fields which correspond to the infinitely many internal reflections. Fortunately an exact knowledge of these secondary fields is not often of practical interest. In optical instruments one tries to reduce these reflections to a minimum absorption since they disturb by "ghost images" the image formation of the instrument.

Further discussion of the treatment of the full steady-state field arising from a dipole source will not be given here, for the purpose of this article is merely to illustrate the general theory of asymptotic development of steady state fields, which will be begun in the next article.

8. Application of Laplace Transform Theory to Maxwell's Equations.

8.1. This article will present some general theory on the solution of Maxwell's equations when the radiation field is due to an arbitrary source. It is this theory which will be applied in the next article to deduce the asymptotic expansion of the steady state field and through this the relation of geometrical optics to the steady state field for any given wave-length.

The essence of this article is one basic theorem. In effect the theorem accomplishes the following. The problem of solving Maxwell's equations for $E(x,y,z,t)$ and $H(x,y,z,t)$ is attacked by introducing the Laplace transforms with respect to time of E and H. These transforms, W and Ω which are independent of time are shown to satisfy the simpler form of Maxwell's equations which one obtains when the source is harmonic. It is then possible to define two vector functions $\vec{A}$ and $\vec{B}$ in terms of W and Ω, from which E and H can be obtained by simple differentiation.

What the theorem accomplishes is that it furnishes the expressions (8.10) for E and H in terms of integrals in which the variable t enters as e^{rt}. Hence the expressions hold regardless of the character of the source, that is, whether it be discontimuous or continuous. Of course implicitly the character of the source must be involved in the integrals through the Laplace transformation. The usefulness of the theorem will be apparent in the next article.

8.2. We alter the problem of solving Maxwell's equations (2.2) by introducing the Laplace adjoints

$$W(x,y,z,r) = \int_0^\infty E\, e^{-rt}\, dt$$

$$\Omega(x,y,z,r) = \int_0^\infty H\, e^{-rt}\, dt \tag{8.1}$$

$$\Phi(x,y,z,r) = \int_0^\infty F\, e^{rt}\, dt.$$

It is a fact of Laplace transform theory that at any fixed point x,y,z, where E,H, and F are bounded and sectionally smooth in t then W, Ω, and Φ will be analytic functions of the complex parameter $\gamma = \alpha + i\beta$ which are regular in the half plane $\alpha > 0$.

Let us now assume that, at a fixed point x,y,z, one or more of the vectors E, H, or F are discontinuous at the times

$$t_\alpha = \frac{1}{c}\psi_\alpha(x,y,z) .$$

Any one function t_α represents the functional dependence of time on position in space so that as one passes from one point to another t_α will vary gradually. At any fixed point of space there may be many functions ψ_α which represent different discontinuities, resulting perhaps from many separate wave fronts passing that point at different times t_α , the wave-fronts themselves resulting from reflections from different obstacles in the medium. It is also possible to regard the many functions ψ_α as branches of some implicit function, $\phi(x,y,z,t) = 0$.

The hypersurfaces $\psi_\alpha(x,y,z) = ct$ divide the x,y,z,t space into separate regions in which E, H, or F are sectionally smooth. From the discontinuity conditions (3.9) which apply to any hypersurface $\phi(x,y,z,t) = 0$ it follows that E,H, and F must satisfy on $\phi \equiv \psi(x,y,z) - ct = 0$, the conditions

$$\begin{aligned} &\operatorname{grad}\psi_\alpha \times [H] + [\epsilon E] = -[F] \\ (8.2)\qquad & \\ &\operatorname{grad}\psi_\alpha \times [E] - [\mu H] = 0. \end{aligned}$$

If in particular ϵ , μ and F are continuous on such a hypersurface $\psi_\alpha = ct$ then it follows by the argument used to obtain (4.15) that $\psi_\alpha(x,y,z)$ must be a solution of the eiconal equations $\psi_x^2 + \psi_y^2 + \psi_z^2 = \epsilon\mu$, i.e., $\psi_\alpha = ct$ must be a set of wave fronts. In addition to such wave fronts other hyper surfaces may occur which are caused by the discontinuities of ϵ , μ or of the vector field F.

We now consider a fixed point x,y,z at which ϵ, μ and σ are continuous, that is, we exclude, for example, points on a refracting surface. By applying integration by parts to relations (8.1) we obtain[9].

$$\begin{aligned} \epsilon\gamma W &= \sum_\alpha [\epsilon E]_\alpha \, e^{-\gamma t_\alpha} + \epsilon\int_0^\infty E_t \, e^{-\gamma t}\, dt \\ (8.3)\qquad \mu\gamma\Omega &= \sum_\alpha [\mu H]_\alpha \, e^{-\gamma t_\alpha} + \mu\int_0^\infty H_t \, e^{-\gamma t}\, dt \\ \gamma\Phi &= \sum_\alpha [F]_\alpha \, e^{-\gamma t_\alpha} + \int_0^\infty F_t \, e^{-\gamma t}\, dt . \end{aligned}$$

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[9] To obtain this result we must regard the integral from 0 to ∞ in (8.1) as the sum of the integrals from 0 to t_1 , from t_1 to t_2 , etc., and lastly, the integral from t_n to ∞. The t_α are the values of t at which E,H and F are discontinuous at a given point (x,y,z). Alternatively, we could introduce the Stieltjes integral in (8.3) and this result would follow at once.

We now return to (8.1) and regard it as the sum indicated in footnote 9. By differentiating under the integral sign in (8.1) and remembering that the discontinuities in E and H are normal to $\psi_\alpha(x,y,z)$ we get,

$$\text{(8.4)}\qquad \begin{aligned} \operatorname{curl} W &= -\frac{1}{c}\sum_{\alpha} \operatorname{grad}\psi_\alpha \times [E]_\alpha e^{-\gamma t_\alpha} + \int_0^\infty \operatorname{curl} E\, e^{-\gamma t}\, dt. \\ \operatorname{curl}\Omega &= -\frac{1}{c}\sum_{\alpha} \operatorname{grad}\psi_\alpha \times [H]_\alpha e^{-\gamma t_\alpha} + \int_0^\infty \operatorname{curl} H\, e^{-\gamma t}\, dt. \end{aligned}$$

Thus it follows by Maxwell's equations (2.2) and the discontinuity relations (8.2) that W and Ω satisfy the equations

$$\text{(8.5)}\qquad \begin{aligned} &\operatorname{curl}\Omega - \frac{\gamma}{c}\epsilon W - \frac{4\pi}{c}\sigma W = \frac{1}{c}\gamma\Phi \\ &\operatorname{curl} W + \frac{\gamma}{c}\mu\Omega = 0. \end{aligned}$$

The Laplace adjoints W and Ω of E and H thus must be solutions of these equations which, at any fixed point x,y,z, are analytic functions of the complex parameter γ and are regular with half plane $R\{\gamma\} > 0$.

The continuity of the vectors $\operatorname{grad}\phi \times E$ and $\operatorname{grad}\phi \times H$ on any surface $\phi(x,y,z) = 0$ of refraction yields the corresponding continuity conditions

$$\text{(8.6)}\qquad \operatorname{grad}\phi \times [W] = \operatorname{grad}\phi \times [\Omega] = 0$$

on any surface. This follows by crossing both sides of (8.1) by $\operatorname{grad}\phi$ and putting $\operatorname{grad}\phi$ under the integral sign on the right side.

Under sufficiently restrictive conditions on E and H it is possible to invert the Laplace transforms by means of the usual inversion theorem, namely,

$$\text{(8.7)}\qquad \begin{aligned} E &= \frac{1}{2\pi i}\int_L W e^{\gamma t}\, d\gamma \\ H &= \frac{1}{2\pi i}\int_L \Omega e^{\gamma t}\, d\gamma \\ F &= \frac{1}{2\pi i}\int_L \Phi e^{\gamma t}\, d\gamma \end{aligned}$$

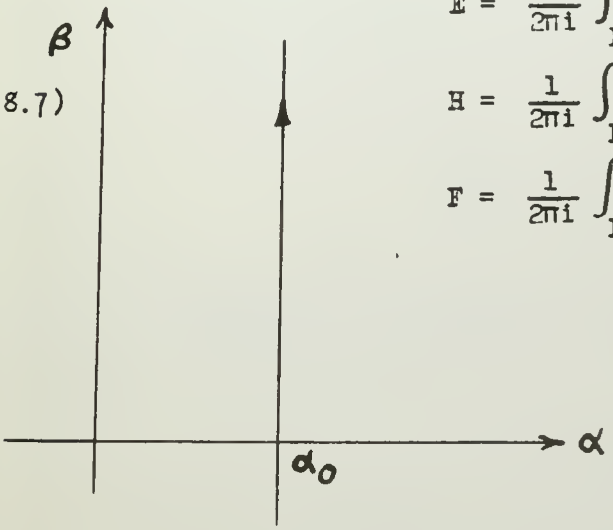

Fig. 13

The path of integration, L, can be any straight line $\alpha = \alpha_0 > 0$ parallel to the imaginary axis. However, we shall not impose such conditions but shall deal directly with the functions W, Ω and Φ.

8.3. The problem of constructing solutions of the original radiation problem (2.2) can be replaced by the problem of finding solutions of the equations (8.5) which satisfy the boundary conditions (8.6) and regularity conditions in the half plane $R\left\{\gamma\right\} > 0$. Indeed we have the following Theorem.

1) Let $\vec{F}(x,y,z,t)$ be a sectionally smooth and uniformly bounded vector field; i.e., $|\vec{F}| < C$ for all x,y,z,t; also let Φ as defined by (8.1) be its Laplace adjoint.

2) Let W and Ω be solutions of the differential equations (8.5) which satisfy the boundary conditions (8.6) and the condition that they be continuous and have continuous derivatives in any closed subdomain D of the x,y,z space where ϵ, μ, and σ are continuous.

3) It is further assumed that to any such subdomain D of the x,y,z space there exists a positive function $M(\rho)$ such that $\int_0^\infty M(\rho)\, d\rho$ exists and such that

$$\begin{gathered} \left|\frac{W}{\gamma^2}\right| \le M(|\beta|) \\ \left|\frac{\Omega}{\gamma^2}\right| \le M(|\beta|) \end{gathered} \tag{8.8}$$

in D and for all $\gamma = \alpha + i\beta$ in the half-plane $\alpha \geq \delta > 0$.

4) Let the integrals $\vec{A}$ and $\vec{B}$, defined by

$$\begin{gathered} \vec{A} = \frac{1}{2\pi i}\int_L \frac{W}{\gamma^2}\, e^{\gamma t}\, d\gamma \\ \vec{B} = \frac{1}{2\pi i}\int_L \frac{\Omega}{\gamma^2}\, e^{\gamma t}\, d\gamma \end{gathered} \tag{8.9}$$

represent two vector fields which have continuous first derivatives and sectionally continuous second and third derivatives in D and for $t \geq 0$.

It follows that the vector fields

$$E = \frac{\partial^2 A}{\partial t^2} \quad \text{and} \quad H = \frac{\partial^2 B}{\partial t^2} \tag{8.10}$$

satisfy Maxwell's equations (2.2), the discontinuity conditions (8.2), the conditions, $\operatorname{grad}\phi \times [\vec{E}] = \operatorname{grad}\phi \times [\vec{H}] = 0$ on the refracting surface $\phi(x,y,z) = 0$, and E and H are zero for $t < 0$.

Proof. Consider a closed domain D of the x,y,z space in which ϵ, μ, σ are continuous and assume that a part of its boundary surface coincides with a surface of refraction. Our assumption about W and Ω means that both vector fields attain definite limits if a point P of Γ is approached. Since the integrals (8.9) are uniformly convergent in D and for $t \geqq 0$ by the hypothesis of the theorem, it follows that also $\vec{A}$ and $\vec{B}$ attain definite limits given by the integrals (8.9). As in the proof of (8.6) it now follows that the condition grad$\phi \times [\vec{A}]$= grad$\phi \times [\vec{B}]$=0 then is a direct consequence of the corresponding conditions for W and Ω on Γ The existence of the second time derivatives of $\vec{A}$ and $\vec{B}$ and the interchange of order of differentiation in (8.10) leads to the conclusion that grad$\phi \times [E]$ = grad$\phi \times [H]$ = 0.

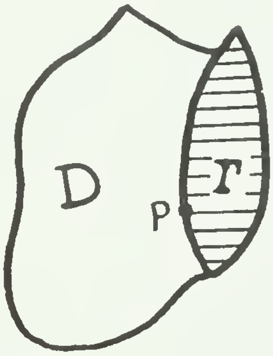

Fig. 14

We next conclude from (8.8) and (8.9) that

(8.11)
$$|\vec{A}| \leqq \frac{e^{\alpha t}}{\pi}\int_0^\infty M d\beta$$
$$|\vec{B}| \leqq \frac{e^{\alpha t}}{\pi}\int_0^\infty M d\beta$$

where $\alpha \geqq \delta > 0$ is the arbitrary abscissa of the path of integration, L. If $t < 0$ then $|\vec{A}|$ and $|\vec{B}|$ are smaller than an arbitrary small number i.e. $A = B = 0$. Hence by (8.10) $E = H = 0$ for $t < 0$.

For the proof that the Maxwell differential equations (2.2) are satisfied note that the hypothesis

$$|\vec{F}| < c$$

implies for the Laplace adjoint Φ that

$$|\Phi| \leqq c\int_0^\infty e^{-\alpha t}\, dt = \frac{c}{\alpha} ,$$

i.e., Φ is uniformly bounded in half plane $\alpha \geqq \delta > 0$. It also follows from (8.7) and a basic theorem on the Laplace transformation that the Laplace adjoints of the iterated integrals

$$\int_0^t F(\tau)\, d\tau \quad \text{and} \quad \int_0^t d\tau \int_0^\tau F(\tau_1)\, d\tau_1$$

are given by the expressions $\frac{\Phi}{\gamma}$ and $\frac{\Phi}{\gamma^2}$ and hence by Laplace's inversion theorem that

$$\int_0^t F(\tau)\, d\tau = \frac{1}{2\pi i} \int_L \frac{\Phi}{\gamma} e^{\gamma t}\, d\gamma \tag{8.12}$$

and

$$\int_0^t d\tau \int_0^\tau F(\tau_1)\, d\tau_1 = \frac{1}{2\pi i} \int_L \frac{\Phi}{\gamma^2} e^{\gamma t}\, d\gamma$$

Replacing the left side of this second equation by an equivalent expression gives:

$$\int_0^t (t-\tau) F(\tau)\, d\tau = \frac{1}{2\pi i} \int_L \frac{\Phi}{\gamma^2} e^{\gamma t}\, d\gamma \;. \tag{8.13}$$

We now consider the vector fields $\vec{A}_1$ and $\vec{B}_1$ defined by

$$\begin{aligned} \vec{A}_1 &= \frac{1}{2\pi i} \int_L \frac{W}{\gamma^3} e^{\gamma t}\, d\gamma \\ \vec{B}_1 &= \frac{1}{2\pi i} \int_L \frac{\Omega}{\gamma^3} e^{\gamma t}\, d\gamma \;. \end{aligned} \tag{8.14}$$

Since by formal differential with respect to t uniformly convergent integrals are obtained we conclude that

$$\begin{aligned} \vec{A} &= \frac{\partial \vec{A}_1}{\partial t} \\ \vec{B} &= \frac{\partial \vec{B}_1}{\partial t} \;. \end{aligned} \tag{8.15}$$

Moreover if in (8.5) we divide by γ^3, multiply by $e^{\gamma t}$, and integrate we obtain

$$\begin{aligned} \int \frac{\text{curl}\, \Omega}{\gamma^3} e^{\gamma t}\, d\gamma &= \frac{\epsilon}{c} \int \frac{W}{\gamma^2} e^{\gamma t} d\gamma + \frac{4\pi}{c} \sigma \int \frac{W}{\gamma^3} e^{\gamma t} d\gamma + \frac{1}{c} \int \frac{\Phi}{\gamma^2} e^{\gamma t}\, d\gamma \\ \int \frac{\text{curl}\, W}{\gamma^3} e^{\gamma t}\, d\gamma &= -\frac{\mu}{c} \int \frac{\Omega}{\gamma^2} e^{\gamma t}\, d\gamma \;, \end{aligned} \tag{8.16}$$

i.e., the curl operation under the integral sign yields uniformly convergent integrals. Hence by (8.14)

$$\text{curl}\, \vec{A}_1 = \frac{1}{2\pi i} \int_L \frac{\text{curl}\, W}{\gamma^3} e^{\gamma t}\, d\gamma$$

$$\text{curl}\, \vec{B}_1 = \frac{1}{2\pi i} \int_L \frac{\text{curl}\, \Omega}{\gamma^3} e^{\gamma t}\, d\gamma$$

and by (8.15), (8.16) and (8.13) we have

$$\begin{aligned} \text{curl}\, \vec{B}_1 - \frac{\epsilon}{c} \frac{\partial \vec{A}_1}{\partial t} - \frac{4\pi}{c} \sigma \vec{A}_1 &= \frac{1}{c} \int_0^t (t-\tau) F(\tau)\, d\tau \\ \text{curl}\, \vec{A}_1 + \frac{\mu}{c} \frac{\partial \vec{B}_1}{\partial t} &= 0. \end{aligned} \tag{8.17}$$

In any domain of the x,y,z,t space where $\vec{A}$ and $\vec{B}$ have continuous time derivatives to the third order we may differentiate (8.17) three times with regard to t. We obtain immediately the desired differential equations (2.2) for $E = \frac{\partial^2 \vec{A}}{\partial t^2}$ and $H = \frac{\partial^2 \vec{B}}{\partial t^2}$.

In order to verify the discontinuity conditions (8.2) let us differentiate (8.17) twice with respect to t. It follows that

$$\text{curl}\ \frac{\partial \vec{B}}{\partial t} - \frac{\epsilon}{c} E - \frac{4\pi}{c} \sigma \frac{\partial \vec{A}}{\partial t} = \frac{1}{c} F$$

(8.18)

$$\text{curl}\ \frac{\partial \vec{A}}{\partial t} + \frac{\mu}{c} H = 0.$$

Let us now assume that $E = \frac{\partial^2 \vec{A}}{\partial t^2}$ and $H = \frac{\partial^2 \vec{B}}{\partial t^2}$ are discontinuous on a hyper surface $\psi(x,y,z) = ct$, which lies in the domain D for $t > 0$. Since $\frac{\partial \vec{A}}{\partial t}$ and $\frac{\partial \vec{B}}{\partial t}$ are continuous in D by hypothesis, and the equations (8.18) are satisfied on both sides of the surface it follows from (8.18) that

$$\left[\text{curl}\ \frac{\partial \vec{B}}{\partial t}\right] - \frac{\epsilon}{c}[E] = \frac{1}{c}[F]$$

(8.19)

$$\left[\text{curl}\ \frac{\partial \vec{A}}{\partial t}\right] + \frac{\mu}{c}[H] = 0,$$

where curl $\vec{B}_t$ means curl $\vec{B}_t(x,y,z,t)$. Let $\vec{B}^*$ be the value of $\vec{B}$ on the surface, that is, $\vec{B}^* = \vec{B}\left(x,y,z, \frac{\psi}{c}(x,y,z)\right)$. Let $\frac{\partial \vec{B}^*}{\partial t} = \vec{B}^*_t$ mean $\frac{\partial \vec{B}}{\partial t}$ and substitute $\frac{\psi}{c}(x,y,z)$ for t after differentiation. Then, for example,

$$\frac{\partial}{\partial y} \vec{B}^*_t = \frac{\partial}{\partial y} \vec{B}_t + \vec{B}_{tt} \frac{\psi_y}{c} .$$

From this it follows that

$$\text{curl}\left[\frac{\partial \vec{B}^*}{\partial t}\right] = \left[\text{curl}\ \frac{\partial \vec{B}}{\partial t}\right] + \frac{1}{c}\ \text{grad}\,\psi \times \left[\frac{\partial^2 \vec{B}}{\partial t^2}\right]$$

(8.20)

$$\text{curl}\left[\frac{\partial \vec{A}^*}{\partial t}\right] = \left[\text{curl}\ \frac{\partial \vec{A}}{\partial t}\right] + \frac{1}{c}\ \text{grad}\,\psi \times \left[\frac{\partial^2 \vec{A}}{\partial t^2}\right]$$

where $\frac{\psi}{c}$ is substituted for t on the right side after differentiation and hence the two equalities hold only on the surface $\psi = ct$. The left sides are zero since

$\left[\frac{\partial \vec{A}^*}{\partial t}\right] = \left[\frac{\partial \vec{B}^*}{\partial t}\right] = 0$; hence, by the definitions of E and H and by (8.19) and (8.20),

$$\operatorname{grad}\psi \times [H] + \epsilon[E] = - \quad [F]$$

(8.21)

$$\operatorname{grad}\psi \times [E] - \mu[H] = 0$$

and these are the conditions (8.2) if we represent the hyper surface in the form $\phi \equiv \psi - ct = 0$. This completes the proof of the theorem.

9. The Asymptotic Development of the Steady State Field.

9.1. This article will employ the theorem just established to arrive at an asymptotic expansion of the steady state field created by harmonically oscillating charges. The expansion to be derived is a series in which the variable is the wave-length λ or $\frac{1}{w}$ where $w = 2\pi$ times the frequency. The coefficients of the powers of λ have the following meaning. One defines first the pulse solution of Maxwell's equations. This is the field which is set up by a source which is 0 for $t < 0$ and which for $t \geq 0$ is 1 in amplitude; that is, at $t = 0$ a charge having the same spatial distribution as the harmonically oscillating charge is suddenly set up. The resulting field satisfying Maxwell's equation is the pulse field for this charge and is denoted by E_0 and H_0. At any point in space this field has discontinuities with respect to the time variable, the discontinuities arising either from the distribution of the source, from the passage by the point of the wave-front emanating from the source, or from discontinuities in ϵ, μ, and σ which cause reflected wave fronts to pass the point.

The essential parts of coefficients of the terms in the asymptotic expansion for u, the spatial component of the steady state E, are respectively, the sum of the discontinuities of E_0, the sum of the discontinuities in the first derivative of E_0, the sum of the discontinuities in the second derivative of E_0, etc. The same statements apply to v, the spatial factor of H, and H_0.

The significance of this result which is embodied in formula (9.18) will be commented upon at greater length at the end of the article. It will be seen shortly that the theory of this article generalizes the result noted for the dipole, namely, that for small wave-lengths the amplitudes of the steady state electromagnetic field obey the same mathematical laws as do the amplitudes of the vectors on a progressing wave-front.

9.2 Let us consider vector fields $\vec{F}$ of the special form

$$\vec{F} = \vec{g}(x,y,z)\, f(t) \tag{9.1}$$

where $\vec{g}(x,y,z)$ is a given vector field and $f(t)$ a given scalar function. We do not exclude the case that the components of $\vec{g}$ and the function $f(t)$ are complex functions of their argument. We assume however that $\vec{g}$ is sectionally smooth and uniformly bounded in the x,y,z space. Similarly $f(t)$ shall be a sectionally smooth and uniformly bounded function of t. Moreover, it shall satisfy the condition

$$\gamma \int_0^\infty f(t)\, e^{-\gamma t}\, dt \leq C(\delta) \tag{9.2}$$

in any half plane $R\{\gamma\} \geq \delta > 0$.

By equation (8.5) of the preceding article the Laplace adjoints of the solutions E and H of the radiation problem of article 2 satisfy the differential equations.

$$\begin{aligned} &\operatorname{curl} \Omega - \frac{\gamma}{c} \epsilon W - \frac{4\pi}{c} \sigma W = \frac{1}{c} \vec{g}\, \gamma \int_0^\infty f(t)\, e^{-\gamma t}\, dt \\ &\operatorname{curl} W + \frac{\mu}{c} \gamma \Omega = 0. \end{aligned} \tag{9.3}$$

9.3 Consider in particular the case

$$f(t) = \eta(t) = \begin{cases} 0, & t < 0 \\ 1 & t > 0. \end{cases} \tag{9.4}$$

This means physically (see article 2.1) that, at the time t = 0, suddenly a charge distribution $\rho = -\operatorname{div} \vec{g}$ is created in the space. This act generates an electromagnetic field $E_0(x,y,z,t)$, $H_0(x,y,z,t)$ which with $t \to \infty$ must approach stationary values, namely the stationary electromagnetic field of a charge distribution In particular $H_0(x,y,z,t) \to 0$. Let us call this particular solution E_0, H_0 the pulse solution of the medium and of the charge distribution $\rho = -\operatorname{div} \vec{g}$.

For this f(t), the right side of the first equation (9.3) is $\frac{\vec{g}}{c}$. Let us now assume that the associated equations for the Laplace adjoints W_0 and Ω_0 of E_0 and H_0, respectively, namely,

$$\begin{aligned} &\operatorname{curl} \Omega_0 - \frac{\gamma}{c} \epsilon W_0 - \frac{4\pi}{c} \sigma W_0 = \frac{\vec{g}}{c} \\ &\operatorname{curl} W_0 + \frac{\gamma}{c} \mu \Omega_0 = 0 \end{aligned} \tag{9.5}$$

have a solution Ω_o, W_o which satisfies the premises of the general theorem of article 8. Then by the theorem of that article (see formulas 8.10) E_o and H_o are given by the integrals

$$(9.6)\qquad \begin{aligned} E_o &= \frac{\partial^2}{\partial t^2} \frac{1}{2\pi i} \int_L \frac{W_o}{\gamma^2} e^{\gamma t}\, d\gamma \\ H_o &= \frac{\partial^2}{\partial t^2} \frac{1}{2\pi i} \int_L \frac{\Omega_o}{\gamma^2} e^{\gamma t}\, d\gamma \end{aligned}$$

That this E_o and H_o are identical with the pulse solution follows from a uniqueness theorem on the solutions of Maxwell's equations.

Also, since W_o is the Laplace adjoint of E_o and Ω_o, the adjoint of H_o, it follows by a basic theorem of Laplace transform theory[10] that

$$(9.7)\qquad \begin{aligned} \frac{1}{2\pi i} \int_L \frac{W_o}{\gamma^2} e^{\gamma t}\, d\gamma &= \int_0^t (t-\tau)\, E_o(\tau)\, d\tau \\ \frac{1}{2\pi i} \int_L \frac{\Omega_o}{\gamma^2} e^{\gamma t}\, d\gamma &= \int_0^t (t-\tau)\, H_o(\tau)\, d\tau \end{aligned}$$

This result tells us what the inverse Laplace transforms of $\frac{W_2}{\gamma^2}$ and $\frac{\Omega_o}{\gamma^2}$ are.

9.4 Consider next a more general function $f(t)$ which satisfies the conditions (9.2). In view of (9.5) a solution W, Ω of the associated equations (9.3) is immediately given by

$$(9.8)\qquad \begin{aligned} W &= \gamma W_o \int_0^\infty f(t)\, e^{-\gamma t}\, dt \\ \Omega &= \gamma \Omega \int_0^\infty f(t)\, e^{-\gamma t}\, dt \;. \end{aligned}$$

On account of (9.2) this solution satisfies the premises of the theorem of the preceding article. By (8.10) and (9.8) the electromagnetic field E, H which belongs to $F = g\, f(t)$ thus is given by the integrals

$$(9.9)\qquad \begin{aligned} E(t) &= \frac{\partial^2}{\partial t^2} \frac{1}{2\pi i} \int_L \left[\frac{W_o}{\gamma} \int_0^\infty f(\tau)\, e^{-\gamma\tau}\, d\tau\, e^{\gamma t} \right] d\gamma \\ H(t) &= \frac{\partial^2}{\partial t^2} \frac{1}{2\pi i} \int_L \left[\frac{\Omega_o}{\gamma} \int_0^\infty f(\tau)\, e^{-\gamma\tau}\, d\tau\, e^{\gamma t} \right] d\gamma \;. \end{aligned}$$

---

[10] The theorem applied is the "Faltung" theorem. It can be stated thus: If $\int_0^\infty F_1(t) e^{-\gamma t}\, dt = G(\gamma)$ and $\int_0^\infty F_2(t) e^{-\gamma t}\, dt = G_2(\gamma)$ then $\frac{1}{2\pi i} \int_L G_1(\gamma) G_2(\gamma) e^{\gamma t}\, d\gamma = \int_0^t F_1(y)\, F_2(t-y)\, dy$. In our application $F_1(t) = E_o$ and $F_2(t) = t$. Then $G_1(\gamma) = W_o$ and $G_2(\gamma) = \frac{1}{\gamma^2}$.

The uniform convergence of these integrals makes it possible to rewrite these integrals as follows:

$$(9.10) \quad \begin{aligned} E &= \frac{\partial^3}{\partial t^3} \frac{1}{2\pi i} \int_L \left[\frac{W_o}{\gamma^2} \int_o^\infty f(\tau)\, e^{-\gamma\tau}\, d\tau\, e^{\gamma t} \right] d\gamma \\ H &= \frac{\partial^3}{\partial t^3} \frac{1}{2\pi i} \int_L \left[\frac{\Omega_o}{\gamma^2} \int_o^\infty f(\tau)\, e^{-\gamma\tau}\, d\tau\, e^{\gamma t} \right] d\gamma\ . \end{aligned}$$

It is readily seen that the new multiple integrals converge uniformly in γ and t and thus the order of integration can be interchanged; that is, we shall integrate with respect to γ first. If we rewrite τ as t and again apply the Faltung theorem to the functions f(t) and $\int_o^t (t-\tau)\, E_o(\tau)\, d\tau$ we obtain through use of formula (9.7),

$$(9.11) \quad \begin{aligned} E &= \frac{\partial^3}{\partial t^3} \int_o^t f(\tau) \int_o^{t-\tau} (t-\tau-\tau_1) E_o(\tau_1)\, d\tau_1\, d\tau \\ H &= \frac{\partial^3}{\partial t^3} \int_o^t f(\tau) \int_o^{t-\tau} (t-\tau-\tau_1)\, H_o(\tau_1)\, d\tau_1\, d\tau \end{aligned}$$

and finally by carrying out two of the differentiations, treating t as the parameter and applying the theorem for the differentiation of an integral with respect to a parameter, we get,

$$(9.12) \quad \begin{aligned} E &= \frac{\partial}{\partial t} \int_o^t f(\tau)\, E_o(t-\tau)\, d\tau \\ H &= \frac{\partial}{\partial t} \int_o^t f(\tau)\, H_o(t-\tau)\, d\tau\ . \end{aligned}$$

This simple result allows us to determine the solution E, H for any source of the type $\vec{F} = \vec{g}(x,y,z)\ f(t)$ if the pulse solution for the case $\vec{F} = \vec{g}(x,y,z)\ \eta(t)$ is known. The formulae (9.12) are known as Duhamel's Principle in Heaviside's operational calculus.

9.5 Duhamel's principle establishes in particular a relationship between the pulse solution E_o, H_o and the solution of the steady state. In fact consider a harmonic oscillator

$$(9.13) \quad \vec{F} = \vec{g}(x,y,z)\ e^{-i\omega t}\ .$$

so that $f(t) = e^{-i\omega t}$. By making the change of variable τ for $t - \tau$ in formula (9.12) and using $f(t) = e^{-i\omega t}$ we find that the electromagnetic field associated

with this oscillator is

$$E = \frac{\partial}{\partial t}\, e^{-i\omega t} \int_0^t E_0(\tau)\, e^{i\omega\tau}\, d\tau$$

$$H = \frac{\partial}{\partial t}\, e^{-i\omega t} \int_0^t H_0(\tau) e^{i\omega\tau}\, d\tau$$

or, after a slight transformation

$$E = E_0(t) - E_0(\infty) + e^{-i\omega t}\Big[E_0(\infty) - i\omega\int_0^t (E_0 - E_0(\infty))e^{i\omega\tau}\, d\tau\Big]$$

(9.14)

$$H = H_0(t) - H_0(\infty) + e^{-i\omega t}\Big[H_0(\infty) - i\omega\int_0^t (H_0 - H_0(\infty))\, e^{i\omega\tau}\, d\tau\Big].$$

This result makes it evident that, with increasing time, E and H assume the form $E = u\, e^{-i\omega t}$, $H = v\, e^{-i\omega t}$ where

$$u(x,y,z) = E_0(\infty) - i\,\omega\int_0^\infty (E_0 - E_0(\infty))e^{i\omega\tau} d\tau$$

(9.15)

$$v(x,y,z) = H_0(\infty) - i\omega\int_0^\infty (H_0 - H_0(\infty))\, e^{i\omega\tau} d\tau \;.$$

In these formulas E_0 and H_0 are functions of the suppressed variables x,y, and z.

9.6 The formulae (9.15) establish a simple relationship between the amplitudes u,v of a time periodic electromagnetic field and the field E_0, H_0 of a sudden signal. We use our result to determine the asymptotic behavior of u and v for large values of ω. We integrate by parts and remember that E_0 and H_0 are discontinuous at the hyper surfaces $t = \frac{1}{c}\psi_\alpha(x,y,z)$. Because of the discontinuities the quantity $E_0 e^{i\omega\tau}\Big|_0^\infty$ equals $-\sum_\alpha [E_0]_\alpha\, e^{ik\psi_\alpha} - E_0(0)$. The full result is

$$u = E_0(x,y,z,0) + \sum_\alpha [E_0]_\alpha\, e^{ik\psi_\alpha} + \int_0^\infty E_{0\tau}\, e^{i\omega\tau}\, d\tau$$

(9.16)

$$v = H_0(x,y,z,0) + \sum_\alpha [H_0]_\alpha\, e^{ik\psi_\alpha} + \int_0^\infty H_{0\tau}\, e^{i\omega\tau}\, d\tau$$

Now, from (3.11) it follows that

$$E_0(x,y,z,0) = -\frac{1}{\epsilon} F(0) = -\frac{1}{\epsilon}\vec{g}(x,y,z)$$

$$H_0(x,y,z,0) = 0.$$

This yields the result:

$$(9.17)\qquad \begin{aligned} u &= -\frac{1}{\epsilon}\vec{g} + \sum_{\alpha}[E_0]_{\alpha}\, e^{ik\psi_{\alpha}} + \int_0^{\infty} E_{0\tau}\, e^{i\omega\tau}\, d\tau \\ v &= \sum_{\alpha}[H_0]_{\alpha}\, e^{ik\psi_{\alpha}} + \int_0^{\infty} H_{0\tau}\, e^{i\omega\tau}\, d\tau \end{aligned}$$

If also the derivatives $\frac{\partial E_0}{\partial t}$ and $\frac{\partial H_0}{\partial t}$ are sectionally smooth, we may again integrate by parts and obtain the representation

$$(9.18)\qquad \begin{aligned} u &= -\frac{1}{\epsilon}\vec{g} + [E_0]_{\alpha}\, e^{ik\psi_{\alpha}} - \frac{1}{i\omega}\sum_{\alpha}\left[\frac{\partial E_0}{\partial t}\right]_{\alpha} e^{ik\psi_{\alpha}} - \frac{1}{i\omega}\int_0^{\infty}\frac{\partial^2 E_0}{\partial\tau^2}\, e^{i\omega\tau} d\tau \\ v &= +\sum_{\alpha}[H_0]_{\alpha}\, e^{ik\psi_{\alpha}} - \frac{1}{i\omega}\sum_{\alpha}\left[\frac{\partial H_0}{\partial t}\right]_{\alpha} e^{ik\psi_{\alpha}} - \frac{1}{i\omega}\int_0^{\infty}\frac{\partial^2 H_0}{\partial\tau^2}\, e^{i\omega\tau} d\tau \end{aligned}$$

Formula (9.18) is of course a complex vector equation. Moreover it is understood that the left and right sides are functions of x,y, and z. The formula shows that the main contribution to u,v for large values of ω is given by the expression

$$(9.19)\qquad \begin{aligned} u &= -\frac{1}{\epsilon}\vec{g} + \sum_{\alpha}[E_0]_{\alpha}\, e^{ik\psi_{\alpha}} \\ v &= +\sum_{\alpha}[H_0]_{\alpha}\, e^{ik\psi_{\alpha}}. \end{aligned}$$

It will be noted that formulas (9.18) give an exact representation for u and v. However, if the successive higher derivatives of E_0 and H_0 are sectionally smooth the integration by parts can be continued and the successive terms of the asymptotic representation obtained. This latter view of the result will prove more useful in application.

At any point in a domain where ϵ, μ, σ are continuous and where there are no sources we have for small λ that

$$(9.20)\qquad \begin{aligned} u &= \sum_{\alpha} U_{\alpha}\, e^{ik\psi_{\alpha}} \\ v &= \sum_{\alpha} V_{\alpha}\, e^{ik\psi_{\alpha}}. \end{aligned}$$

As remarked in article 7 in connection with formulas (7.22) to (7.24) the functions $\psi_\alpha(x,y,z)$, in this case, must be solutions of the equation $\psi_x^2 + \psi_y^2 + \psi_z^2 = \epsilon\mu$, i.e., must be wave fronts. The factors U_α, V_α are given by the discontinuities $U_\alpha = [E_0]_\alpha$; $V_\alpha = [H_0]_\alpha$ of the pulse solution E_0, H_0 on these wave-fronts. These discontinuities thus determine directly the asymptotic behavior of the steady state solution in the case of small wave length.

In general we can use the formulae (9.18) to construct steady state solutions from the particular solution which represents the field of a sudden introduction of electric charge into the space. The formulae make it evident that the higher terms of an asymptotic development of u,v in case of large values of k are given by the discontinuities of the derivatives $\frac{\partial E_0}{\partial t}$, $\frac{\partial^2 E_0}{\partial t^2}$, etc. of the pulse solution.

The basic result (9.18) is useful both theoretically and practically. On the theoretical side it says that if we have full knowledge of the pulse solution of Maxwell's equations due to a particular source, or if we at least know the discontinuities of that pulse solution and its derivatives, we can express the steady state due to a harmonic oscillation of that same source in terms of the discontinuities of the pulse solution and its derivatives. While the analogous principle is known and has been used in circuit theory, it has special significance for electromagnetic theory. As the text points out, at a point x,y,z in a domain in which ϵ, μ, and σ are continuous and in which there are no sources, the discontinuities must arise from wave fronts passing this point. Of these wave-fronts one would be the wave front arriving directly from the source, the transmitted wave-front. The other wave-fronts will arise by reflection from discontinuities in ϵ, μ, and σ, existing at other points in space, and due, for example to the presence of obstacles. Now it was shown in earlier articles that the behavior of E and H on these wave-fronts is the geometrical behavior usually furnished by geometrical optics. Hence the discontinuities of E_0 and H_0 on the wave fronts are precisely what geometrical optics supplies as to the field at the point (x,y,z). In other words, the first terms in u and v of formula (9.18), except for the source, is essentially the geometrical optics

description of the steady state field. The theory shows then that for very small wave lengths, the geometrical optics solution is a good approximation and at least a first approximation to the steady state field.

The result (9.18) while useful in a variety of applications is one formulation of the asymptotic expansion of the steady state field. A new formulation and its advantages will be given in the next article.

On the practical side the asymptotic expansions (9.18) can be used to obtain approximate values for the steady state field. If only the geometrical optics representation of the field is known then the theory tells us at least the true character of the approximation in a much fuller and more explicit manner than hitherto known. If in addition the pulse solution is known several terms of the asymptotic expansion can be computed. It will be shown in a later paper that the pulse solution can be obtained in some diffraction problems. Of course a fixed number of terms of the asymptotic expansion will be a better approximation for smaller wave length.

10. Complex Integral Representation of the Asymptotic Development.

10.1 The asymptotic representation of the steady state electromagnetic field furnished by formula (9.18) is adequate when the derivates of E_0 and H_0, as well as E_0 and H_0 themselves have finite discontinuities. If, however, the discontinuities in E_0 and H_0 have this form but any of their derivatives become infinite at a discontinuity t_α of t, the continued integration by parts cannot be carried out. Such is the case, for example, when the form of E_0 or H_0 near t_α is that of the function z^m for z near 0 where m is fractional. Here continued differentiation of E_0 or H_0 will ultimately produce infinite discontinuities.

To handle these more complicated discontinuities a new form of the steady state field will be introduced which will likewise give rise to an asymptotic expression but which avoids the use of higher derivatives. The new form is embodied in the complex integral representation (10.12). It will be shown by mathematical examples how this representation can be used to obtain asymptotic representations of steady state fields. The first of these examples presents a discontinuity which can be handled by the method of the preceding article; the others cannot. Among these examples is one of considerable

generality, namely, the result incorporated in the series (10.30). The type of discontinuity treated by this series does arise in an application to diffraction which will be treated in a later paper.

10.2 Let us assume in this section that E_o and H_o are sectionally analytic. This means that E_o and H_o possess continuous derivatives of any order in the interior of any domain, D, of the x,y,z,t space which does not contain any points of the wave front hyper surfaces $\psi_\alpha - ct = 0$ or of the surfaces of refraction. This assumption about E_o and H_o is justified if ϵ, μ, σ and the components of the vector field F are sectionally analytic. While E_o and H_o will have finite jump discontinuities at any wave front or surface of refraction, we do not assume that the higher order derivatives of E_o and H_o attain finite limits if one of these surfaces is approached.

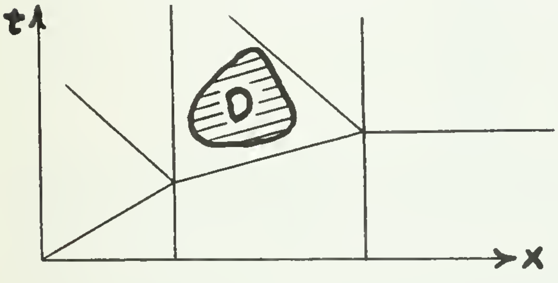

Fig. 15

We write the formulae (9.15) in the following modified form

$$(10.1)\qquad \begin{aligned} u &= \lim_{T\to\infty}\left\{E_o(T)\, e^{i\omega T} - i\omega\int_o^T E_o(t)\, e^{i\omega t}\, dt\right\} \\ g &= \lim_{T\to\infty}\left\{H_o(T)\, e^{i\omega T} - i\omega\int_o^T E_o(t)\, e^{i\omega t}\, dt\right\}. \end{aligned}$$

We next define two vector fields A(x,y,z,t), B(x,y,z,t) by the integrals

$$(10.2)\qquad \begin{aligned} A &= \frac{1}{2\pi i}\int_o^\infty \frac{E_o(\tau)}{(\tau - t)^2}\, d\tau \\ B &= \frac{1}{2\pi i}\int_o^\infty \frac{H_o(\tau)}{(\tau - t)^2}\, d\tau \end{aligned}$$

and interpret $t = t_1 + i\, t_2$ as a complex parameter. Thus A and B are complex functions of the complex variable t (and, of course, of x,y,z, which however are fixed for present purposes).

The properties of these complex functions are best studied by applying a formula for integration by parts which holds for Stieltjes integrals and which thereby permits us to treat $E_0(\tau)$ even though its derivate may have infinite discontimuities on the range 0 to ∞ for τ. According to this formula[11] A and B may be written

$$A(t) = \frac{1}{2\pi i}\int_0^{\infty} \frac{d\,E_0(\tau)}{\tau - t}$$

(10.3)

$$B(t) = \frac{1}{2\pi i}\int_0^{\infty} \frac{d\,H_0(\tau)}{\tau - t}$$

where the integrals are Stieltjes integrals. For $t = t_1 + it_2$ not on the positive real axis, A and B are analytic functions of t. If E_0 and H_0 are sectionally analytic in τ the analytic functions represented by the two integrals when $t_2 \neq 0$ can be continued past the positive real axis along paths which cross the axis at values of t where $E_0(\tau)$ and $H_0(\tau)$ are analytic in the real variable τ. However at values of τ, where E_0 and H_0 are discontinuous, the functions A(t) and B(t) possess singularities. The analytic continuation through values of τ where E_0 and H_0 are sectionally analytic will create multiple-valued functions of A and B one branch of which in the upper half-plane ($t_2 > 0$) is represented by the integral and another branch of which is represented in the lower half-plane by the integral when $t_2 < 0$. Since the singularities of A(t) and B(t) are at those points $t_\alpha = \frac{1}{c}\psi_\alpha$ on the positive t_1-axis where E_0 and H_0 are discontinuous, the stretches of the positive t_1-axis between singularities may be taken to be the branch cuts of the Riemann surfaces for A and B. Also, since the negative t_1-axis is a region of analyticity for A(t) and B(t), one may contimue from $t_2 > 0$ to $t_2 < 0$ across the negative t_1-axis and remain on the same branch of the function. By following such a path one passes from the integral representation of the functions in the upper half-plane to the integral representations in the lower half-plane.

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[11] The formula for integration by parts referred to is proven in Wintner, A: Spektraltheorie, S. Hirzel, Leipzig, 1929. The assertions which follow in this paragraph regarding A and B are all established for the equivalent forms (10.3) in Wintner, where also a full discussion of the Stieltjes integral may be found. Pages 83 to 88 and pp. 91-100 are most relevant background material for this paragraph. Wintner's integrals are defined over the range $-\infty$ to ∞. Since our E_0 and H_0 are 0 for $t < 0$ we may restrict the integrals here to the range 0 to ∞. Some of the material in Wintner may also be found in Widder, D.V.: The Laplace Transform, Princeton University Press, 1941, p. 324 ff.

One can verify readily that the vectors A and B satisfy Maxwell's equations (2.2), namely,

$$\text{(10.4)} \qquad \begin{aligned} \mathrm{curl}\, B - \frac{\epsilon}{c} A_t - \frac{4\pi\sigma}{c} A &= \frac{1}{c} G_t \\ \mathrm{curl}\, A + \frac{\mu}{c} B_t &= 0 \end{aligned}$$

where t is interpreted as a complex parameter and where G is by definition the vector field

$$\text{(10.5)} \qquad G = -\frac{1}{2\pi i}\int_0^\infty \frac{F(\tau)}{(\tau - t)^2}\, d\tau = -\frac{\vec{g}(x,y,z)}{2\pi i t} \;,$$

since for the pulse solution $F(\tau) = \vec{g}(x,y,z)\,\eta(\tau)$ where $\eta(\tau)$ is given by (9.4). Indeed, by differentiating formula (10.2) under the integral sign, remembering that E_0 and H_0 are discontinuous at $\tau = \psi_\alpha$ (cf. the derivation of formula (8.4) we obtain

$$\text{(10.6)} \qquad \begin{aligned} \mathrm{curl}\, A &= -\frac{1}{2\pi i}\sum_\alpha \frac{\mathrm{grad}\,\psi_\alpha \times [E_0]_\alpha}{(\frac{1}{c}\psi_\alpha - t)^2} + \frac{1}{2\pi i}\int_0^\infty \frac{\mathrm{curl}\, E_0}{(\tau - t)^2}\, d\tau \\ \frac{\partial A}{\partial t} &= \frac{1}{2\pi i}\sum_\alpha \frac{[E_0]_\alpha}{(\frac{1}{c}\psi_\alpha - t)^2} + \frac{1}{2\pi i}\int_0^\infty \frac{E_{0\tau}}{(\tau - t)^2}\, d\tau \end{aligned}$$

and similar equations for curl B, $\frac{\partial B}{\partial t}$ and $\frac{\partial G}{\partial t}$.

The equations (10.4) thus are consequences of Maxwell's equations for E_0, H_0 and of the discontinuity conditions (3.9).

10.3 It is possible to invert the integral formulae (10.2) provided that $E_0(t)$ and $H_0(t)$ are sectionally continuous. Let τ, as before, denote real values. Then according to Stieltjes' inversion theorem:[12]

$$\text{(10.7)} \qquad \begin{aligned} \lim_{\epsilon \to 0} \int_0^\tau \Big(A(t_1 + i\epsilon) - A(t_1 - i\epsilon)\Big)\, dt_1 &= E_0(\tau) - E_0(0) \\ \lim_{\epsilon \to 0} \int_0^\tau \Big(B(t_1 + i\epsilon) - B(t_1 - i\epsilon)\Big)\, dt_1 &= H_0(\tau) - H_0(0) \,. \end{aligned}$$

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[12] See Wintner, loc. cit., p. 96 or Widder, loc. cit., p. 339.

If E_0 or H_0 are discontinuous at the point τ then $E_0(\tau)$ and $H_0(\tau)$ are to be replaced by the arithmetic means

$$\text{(10.8)}\qquad \begin{aligned} &\tfrac{1}{2}\left(E_0(\tau+0)+E_0(\tau-0)\right)\\ &\tfrac{1}{2}\left(H_0(\tau+0)+H_0(\tau-0)\right) . \end{aligned}$$

Consider now the first of equations (10.7). We multiply by $i\omega e^{i\omega\tau}$ and integrate with respect to τ from $\tau = 0$ to $\tau = T$. By carrying out the integration under the limit sign and integrating by parts on the left and integrating directly the second term on the right we get

$$\lim_{\epsilon\to 0}\left\{e^{i\omega T}\int_0^T \big(A(\tau+i\epsilon)-A(\tau-i\epsilon)\big)\,d\tau \;-\int_0^T e^{i\omega\tau}\big(A(\tau+i\epsilon)-A(\tau-i\epsilon)\big)\,d\tau\right\}$$

$$= i\omega\int_0^T E_0(\tau)\,e^{i\omega\tau}\,d\tau + E_0(0)\,(1-e^{i\omega T}) .$$

This gives with the aid of (10.7) and (10.1):

$$\text{(10.9)}\qquad \lim_{\epsilon\to 0}\left\{\int_0^\infty e^{i\omega\tau}\big(A(\tau+i\epsilon)-A(\tau-i\epsilon)\big)\,d\tau\right\} = u - E_0(0)$$

The limit on the left side is equal to the integral, $\int_L A(t)\,e^{i\omega t}dt$, where the path of integration, L, is a curve enclosing the positive real axis as illustrated in Fig. 16 . Since E_0 is discontinuous at $\tau = 0$ we must replace $E_0(0)$ in (10.9) by $\frac{1}{2}E_0(+0)$. The latter however by (3.11) is $-\frac{1}{2\epsilon}\vec{g}(x,y,z)$. We get therefore

$$\text{(10.10)}\qquad \begin{aligned} u &= -\frac{1}{2\epsilon}\vec{g} + \int_L A(t)\,e^{i\omega t}\,dt\\ v &= \int_L B(t)\,e^{i\omega t}\,dt. \end{aligned}$$

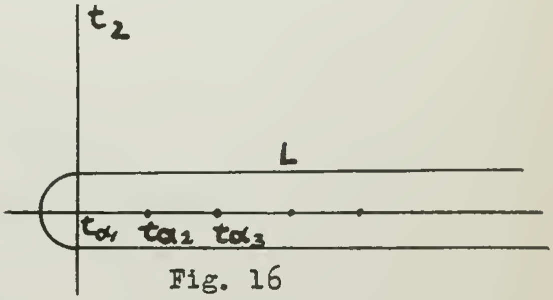

Fig. 16

10.4 We may deform the path of integration L and the cuts along which the branches of the Riemannian surface are joined together. The integrals

(10.10) will not change if, after the deformation, the path L still surrounds all the cuts of the surface.[13] This leads us to Fig. 17. The cuts go from each singularity t_α to $t_\alpha + i\infty$ and the path L surrounds all these cuts. Now consider a point x,y,z in whose neighborhood $\vec{g}(x,y,z) \equiv 0$. Then

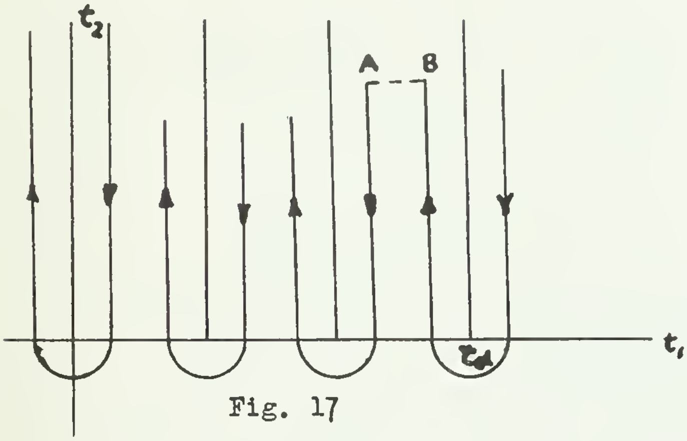

Fig. 17

$$u = \int_L A(t)\, e^{i\omega t}\, dt \tag{10.11}$$

$$v = \int_L B(t)\, e^{i\omega t}\, dt .$$

Since at such an x,y,z E_0, H_0 and all time derivatives are zero at $t = 0$, in this case we conclude that there is no singularity at $t = 0$.

We now contract the different sections of the path L to the straight lines which represent the cuts. We replace the variable of integration t by $\frac{1}{c}\psi + \tau$ and obtain u and v in the form

$$u = \sum_{\alpha=1}^{\infty} U_\alpha\, e^{ik\psi_\alpha}$$

$$v = \sum_{\alpha=1}^{\infty} V_\alpha\, e^{ik\psi_\alpha} \tag{10.12}$$

where

$$U_\alpha = \int_M A\left(\frac{1}{c}\psi_\alpha + \tau\right) e^{i\omega\tau}\, d\tau$$

$$V_\alpha = \int_M B\left(\frac{1}{c}\psi_\alpha + \tau\right) e^{i\omega\tau}\, d\tau .$$

where the path M of integration now surrounds the positive imaginary axis of the complex τ - plane.

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[13] Strictly the path L should first be deformed into finite U shaped paths joined by jorizontal sections such as AB in the figure and then let AB approach infinity. The function $A(t)e^{i\omega t}$ approaches zero however as $t \rightarrow a + i\infty$ for any a. This follows from the fact that A(t) which is given by (10.2) approaches 0 as the imaginary part of $t \rightarrow i\infty$ and so does $e^{i\omega t}$. Hence $\int_{AB} A(t)\, e^{i\omega t}\, dt$ approach zero as AB moves upward.

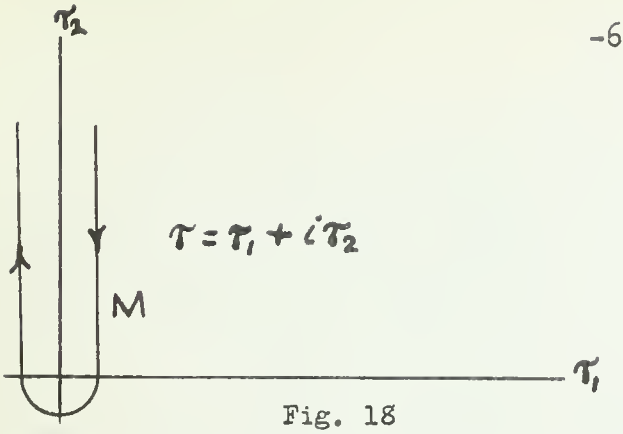

Fig. 18

The result (10.12) is the basic result of this article and furnishes the second method of obtaining the asymptotic expansion of the steady state field. The actual form of the result does not make evident two facts about the expansion more evident in the formulation of article 9. The values of the U_α and V_α, since they are complex integrals over closed paths, will actually depend upon the singularities of the integrands A and B. However, it was pointed out earlier that the singularities of these functions are those of the pulse solution E_0 and H_0. Hence the pulse solution is involved as in the earlier formulation. The other essential feature of the previous asymptotic expansion is that the basic variable is the wave length λ. It will soon appear that each U_α and V_α in formula (10.12) is an infinite series in λ or in $\frac{1}{\omega}$ so that again the final expressions for u and v are asymptotic expansions in λ.

10.5 The individual terms of the series (10.12), namely,

$$u_\alpha = U_\alpha \; e^{ik\psi_\alpha} \qquad (10.13)$$
$$v_\alpha = V_\alpha \; e^{ik\psi_\alpha}$$

are solutions of Maxwell's equations. Indeed

$$\text{curl } V_\alpha = \int_M \left(\text{curl B} + \frac{1}{c}\,\text{grad}\,\psi_\alpha \times \frac{\partial B}{\partial t}\right) e^{i\omega\tau} d\tau$$
$$= \int_M \text{curl B } e^{i\omega\tau} \; d\tau - ik\,\text{grad}\,\psi_\alpha \times V_\alpha .$$

Hence by (10.4) and partial integration, G being 0 at (x,y,z),

$$\text{curl } V_\alpha + ik\,\text{grad}\,\psi_\alpha \times V_\alpha = \frac{4\pi}{c}\sigma U_\alpha - ik\epsilon U_\alpha ,$$
$$(10.14)$$
$$\text{curl } U_\alpha + ik\,\text{grad}\,\psi_\alpha \times U_\alpha = ik\mu V_\alpha ;$$

and hence

$$\text{curl } v_\alpha + ik\epsilon u_\alpha - \frac{4\pi}{c}\sigma u_\alpha = 0,$$
$$(10.15)$$
$$\text{curl} u_\alpha - ik\mu v_\alpha = 0.$$

10.6 In the next few pages we shall illustrate how the general result (10.12) can be applied. The illustrations are, in so far as this paper is concerned, without physical significance. However, in a paper on diffraction now being prepared, it will appear that the second of the illustrations pertains to physical problems. The first illustration deals with an E_0 and H_0 whose discontinuities and the discontinuities of their derivatives are finite jumps and could therefore be treated by the method of article 9. The second illustration must necessarily be treated by the method of this article.

Consider, for example, the case that E_0 and H_0 and all their derivatives approach finite limits if τ approaches 0, that is, if $t = \frac{1}{c}\psi_1(x,y,z) + 0$. Let us also assume that $E_0 = H_0 = 0$ for $t = \frac{1}{c}\psi_1 - 0$ so that $t = \frac{1}{c}\psi_1$ represents the directly transmitted wave front of the pulse solution. We may expand $E_0(t)$ in the neighborhood of $t = \frac{1}{c}\psi_1$ by means of Taylor's theorem for t values greater than $\frac{\psi_1}{c}$. Then

$$(10.16) \qquad E_0(t) = \sum_{\nu=0}^{\infty} \frac{E_0^{\nu}}{\nu!}\left(t - \frac{1}{c}\psi_1\right)^{\nu}, \quad t > \frac{\psi_1}{c}$$

wherein E_0^{ν} means $\frac{\partial^{\nu} E_0}{\partial t^{\nu}}$ at $t = \frac{1}{c}\psi_1 + 0$.

For any $E_0(t)$ the corresponding $A(\tau)$ is given by (10.2). Now the behavior of the function $A(t)$, in the neighborhood of $\tau = 0$ or $t = \frac{1}{c}\psi_1$ is the same as that of the integral

$$(10.17) \qquad A_1(t) = \frac{d}{dt}\,\frac{1}{2\pi i}\int_{\frac{1}{c}\psi_1}^{\delta + \frac{1}{c}\psi_1} \sum_{\nu=0}^{\infty} \frac{E_0^{\nu}}{\nu!}\,\frac{\left(\tau - \frac{1}{c}\psi_1\right)^{\nu}}{(\tau - t)}\, d\tau$$

where δ is arbitrarily small. Since we are only interested in the singular part of $A(t)$ we are permitted to omit regular contributions of the integral. To determine the singular part of expression (10.17), replace $t - \tau$ by ρ, say, and reverse summation and integration. The integral then has the form

$$\int \frac{\left[\left(t - \frac{\psi_1}{c}\right) - \rho\right]^{\nu}}{\rho}\, d\rho$$

Since ν is integral, binomial expansion of the numerator shows that the only singular term will be $\dfrac{(t - \frac{\psi_1}{c})^\nu}{\rho}$. We can integrate this singular term, for the parentheses are independent of ρ. Taking, then, the factor $(t - \frac{\psi_1}{c})^\nu$ out of the integral we may express the singularity of the integral at $t = \psi/c$ merely by $\log(t - \psi_1/c)$.

This result leads to the following expression for the singularity of $A_1(t)$ or of $A(t)$:

$$A_1(t) \approx - \frac{d}{dt} \frac{1}{2\pi i} \sum_{\nu=0}^{\infty} \frac{E_0^\nu}{\nu!} (t - \frac{\psi_1}{c})^\nu \log(t - \frac{\psi_1}{c}) \tag{10.18}$$

We may introduce this expression into the basic result (10.12) without changing the result for U_α because the integral of this regular part around the closed path M of Figure 18 will be zero.[14] It follows since $\tau = t - \psi_1/c$ that

$$U_\alpha = \frac{i\omega}{2\pi i} \sum_{\nu=0}^{\infty} \int_M \frac{E_0^\nu}{\nu!} \tau^\nu \log \tau \, e^{i\omega\tau} \, d\tau \, .$$

Since $\log \tau$ represents the singularity of the term at $\tau = 0$ or at $t = \frac{\psi_1}{c}$ and since the values of $\log \tau$ on the two sides of the branch line differ by the constant $-2\pi i$ we conclude

$$\begin{aligned} U_\alpha &= -i\omega \sum_{\nu=0}^{\infty} \int_0^{i\infty} \frac{E_0^\nu}{\nu!} \tau^\nu e^{i\omega\tau} d\tau \\ &= \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(i\omega)^\nu} E_0^\nu \; ^{15} \end{aligned} \tag{10.19}$$

The same result follows direcly from (9.18). It should be noted that $E_0^\nu = [E_0^\nu]$ since the left hand derivative is 0 in this example.

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[14] The integral $\int_M F(\tau) e^{i\omega\tau} d\tau$ where $F(\tau)$ is regular is zero since the path is from i oo to 0 and then from 0 to $i\infty$.

[15] Integrate $\tau^\nu e^{i\omega\tau} d\tau$ by parts letting $u = \tau^\nu$ and $dv = e^{i\omega\tau} d\tau$. Repeat this ν times; the explicit "uv" term drops out each time leaving finally the integral of $e^{i\omega\tau}$ with a constant multiplier.

We now illustrate the application of the basic result (10.12) with an example in which the method of the preceding article does not apply. We consider the case where

$$E_0 = \sum_{n=0}^{\infty} c_n(t - \frac{1}{c}\psi)^{n + 1/2} \qquad , \text{ if } t \geq \frac{1}{c}\psi$$

$$= 0 \qquad , \text{ if } t \leq \frac{1}{c}\psi .$$

Again A(t) is given by the integral in (10.2), the τ of that formula being equal to $t - \frac{\psi}{c}$ in the representation of E_0 here. The behavior of A(t) in the neighborhood of $t = \frac{1}{c}\psi$ is given by the integral,

$$A_1(t) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} c_n \frac{d}{dt} \int_0^{\delta} \frac{\tau^{n + 1/2}}{\tau - (t - \frac{1}{c}\psi)} \quad d\tau$$

where t here assumes values greater than $\frac{\psi}{c}$. By repetition of the argument used in the preceding examples to obtain formula (10.18) we may state that the singular part of $A_1(t)$ is given by

$$A_1(t) = \frac{d}{dt} \frac{1}{2\pi i} \sum_{n=0}^{\infty} c_n(t - \frac{1}{c}\psi)^n \int_0^{\delta} \frac{\tau^{1/2}}{\tau - (t - \frac{1}{c}\psi)} \quad d\tau. \tag{10.20}$$

Since $\int \frac{1}{\sqrt{\tau}} d\tau$ has no singularity at $\tau = 0$, it follows that the singularity of the integral in $A_1(t)$ is the same as the singularity of the integral

$$\int_0^{\delta} (\frac{\tau^{1/2}}{\tau - (t - \frac{1}{c}\psi)} - \frac{1}{\sqrt{\tau}}) d\tau .$$

By adding fractions and by repeating the reasoning above in connection with formula (10.18) we have that the singularity in this integral is the same as the singularity in

$$(t - \frac{\psi}{c}) \int_0^{\delta} \frac{d\tau}{\sqrt{\tau}[\tau - (t - \frac{\psi}{c})]}$$

By adding and subtracting the integral for the range δ to ∞ we have that this last integral equals

$$(t - \frac{\Psi}{c}) \int_0^\infty \frac{d\tau}{\sqrt{\tau}\,[\tau - (t - \frac{\Psi}{c})]} + \text{Regular terms.}$$

The integral can be evaluated by a standard integral form and we obtain

$$= -\pi i \sqrt{t - \frac{\Psi}{c}} + \text{Regular terms.}$$

By substituting this result in formula (10.20) we obtain that

$$A(t) = -\frac{1}{2} \frac{d}{dt} \sum_{n=0}^{\infty} C_n (t - \frac{\Psi}{c})^{n+1/2} + \text{Regular terms.}$$

We now introduce this value in (10.12) and obtain for U_α :

$$U_\alpha = \sum_{n=0}^{\infty} C_n \int_M - \frac{1}{2} \frac{d}{d\tau} (\tau)^{n+1/2} \quad e^{i\omega\tau} d\tau$$

Now the path M goes from $i\infty$ to 0 on the right side of the branch line and from 0 to $i\infty$ on the left side. Because the quantity $\tau^{n+1/2} = \tau^n \tau^{1/2}$ appears in the integrand, the integrand is multiple-valued by virtue of the $\tau^{1/2}$. On the two sides of the branch line the two values of $\tau^{1/2}$ differ by a negative sign. Hence we may combine the two integrals thus:

$$U_\alpha = \sum_{n=0}^{\infty} C_n \int_0^{i\infty} \frac{d}{d\tau} (\tau)^{n+1/2} \quad e^{i\omega\tau} d\tau \quad .$$

By a change of variable we obtain

$$U_\alpha = \sum_{n=0}^{\infty} \frac{C_n}{(-i\omega)^{n+1/2}} \int_0^\infty \frac{d}{d\bar{\tau}} (\bar{\tau})^{n+1/2} \quad e^{-\bar{\tau}} d\bar{\tau}$$

An integration by parts gives

$$U_\alpha = \sum_{n=0}^{\infty} \frac{C_n}{(-i\omega)^{n+1/2}} \int_0^\infty \bar{\tau}^{n+1/2} \quad e^{-\bar{\tau}} d\bar{\tau}$$

In view of the definition of the Gamma function we get

$$U_\alpha = \sum_{n=0} \frac{C_n}{(-i\omega)^{n+1/2}} \Gamma(n + 3/2) . \tag{10.21}$$

This result shows, as did our preceding example that U_α is an infinite series in the basic variable $\frac{1}{\omega}$ or in λ and it shows further that U_α approaches 0 with λ.

10.7 Our examples show how the singularities of A(t) and B(t) determine the U_α and V_α and hence the u and v. Since the analysis is lengthy it is desirable to obtain a general result on the value of U_α and V_α for a general class of functions A(t) and B(t) and thereby avoid frequent recourse to such analysis. We therefore now shall obtain a general result of this nature. It is this general result which will be applied in a later paper.

Let us assume that $E_o(t)$ and $H_o(t)$ are sectionally analytic. Consider for example $E_o(t)$ in the neighborhood of a point t_o of discontinuity. Let

$$\begin{aligned} E_o &= \alpha(t_o - t) \sum_{n=1}^{\infty} \frac{C_n}{n!}(t_o - t)^n \quad , \text{ for } t < t_o \\ E_o &= \beta(t - t_o) \sum_{n=1}^{\infty} \frac{D_n}{n!}(t - t_o)^n \quad , \text{ for } t > t_o \ . \end{aligned} \tag{10.22}$$

where α and β are functions of $t_o - t$. We shall require that the functions $\alpha(\tau)$ and $\beta(\tau)$ be analytic functions of τ which are not regular at $\tau = 0$, for example algebraic functions of the type τ^k, where k is not necessarily an integer. We also include the case that α or β equals $\log \tau$.

We know that A(t) as defined by (10.2) has the same singularities at $t = t_o$ as the sum of the integrals

$$A_1(t) = \frac{1}{2\pi i}\int_{t_o-\delta}^{t_o} \frac{E_o d\tau}{(\tau - t)^2} \quad , \quad A_2(t) = \frac{1}{2\pi i}\int_{t_o}^{t_o+\delta} \frac{E_o d\tau}{(\tau - t)^2} \tag{10.23}$$

where δ is arbitrarily small.

We consider the terms of the series (10.22) individually, i.e. we consider the function $\alpha(t_o - t)(t_o - t)^n$ for $t < t$ and $\beta(t - t_o)(t - t_o)^n$ for $t > t_o$. The contribution of these terms to the integrals in (10.23) are

$$A_1^n(t) = -\frac{d}{dt}\frac{1}{2\pi i}\int_o^\delta \frac{\alpha(\tau)}{\tau-(t_o-t)}\,\tau^n\,d\tau$$

$$A_2^n(t) = \frac{d}{dt}\frac{1}{2\pi i}\int_o^\delta \frac{\beta(\tau)}{\tau-(t-t_o)}\,\tau^n\,d\tau .$$

By the same argument as that used to derive (10.18) from (10.17) we conclude that A_1^n and A_2^n differ only by regular terms from the expressions

$$A_1^n(t) \approx -\frac{d}{dt}(t_o - t)^n\,\frac{1}{2\pi i}\int_o^\delta \frac{\alpha(\tau)}{\tau-(t_o-t)}\,d\tau$$

(10.24)

$$A_2^n(t) \approx \frac{d}{dt}(t - t_o)^n\,\frac{1}{2\pi i}\int_o^\delta \frac{\beta(\tau)}{\tau-(t-t_o)}\,d\tau .$$

We next determine the contribution of these expressions to the function

(10.25) $$U = \int_M A(t_o + \tau)\,e^{i\omega\tau}\,d\tau .$$

Here U denotes any one of the U_α in (10.12). Letting

$$f(t_o - t) = \frac{1}{2\pi i}\int_o^\delta \frac{\alpha(\tau)}{\tau-(t_o-t)}\,d\tau \quad \text{and}$$

(10.26)

$$g(t - t_o) = \frac{1}{2\pi i}\int_o^\delta \frac{\beta(\tau)}{\tau-(t - t_o}\,d\tau$$

we get from (10.26), (10.24) and (10.25)

$$U_1^n = -i\omega\int_{M'} f(t)t^n\,e^{-i\omega t}\,dt$$

(10.27)

$$U_2^n = -i\omega\int_M g(t)t^n\,e^{i\omega t}\,dt$$

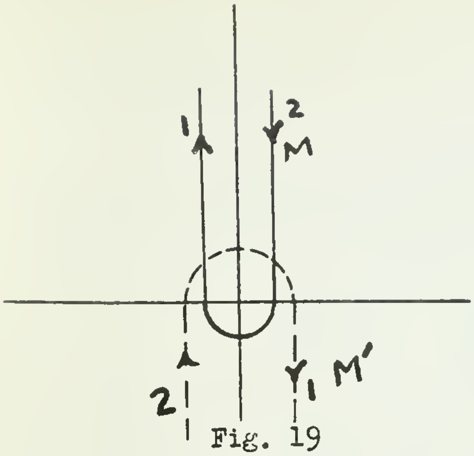

Fig. 19

where M and M' are the paths of integration indicated in Fig. 19. These two paths surround the positive and negative imaginary axis respectively. The two branches of each path lie in different sheets of the Riemannian surfaces of f and g.

By introducing the differences

$$[f] = f_1 - f_2$$

$$[g] = g_1 - g_2$$

in corresponding points of the two paths (Fig. 19) we may transform (10.27) into

$$\text{(10.28)} \quad \begin{aligned} U_1^n &= - i\omega \int_0^{-i\infty} [f]\, t^n e^{-i\omega t}\, dt \\ U_2^n &= i\omega \int_0^{i\infty} [g] t^n e^{i\omega t}\, dt. \end{aligned}$$

We determine the differences $[f]$ and $[g]$ as follows. Consider for example

$$f(t) = \frac{1}{2\pi i} \int_0^{\sigma} \frac{\alpha(\tau)}{\tau - t}\, d\tau .$$

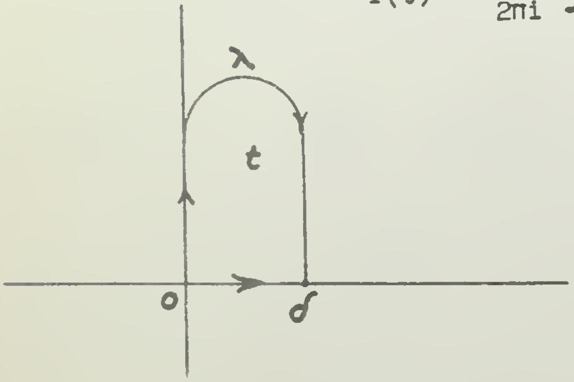

Fig. 20

The integral determines the value $f_1(t)$ in the first sheet of the Riemannian surface. The associated value $f_2(t)$ in the second sheet is obtained by the integral

$$f(t) = \frac{1}{2\pi i} \int_{\lambda} \frac{\alpha(\tau)\, d\tau}{\tau - t}$$

where λ is a curve connecting the points 0 and σ such that the point t lies between the curve λ and the real axis. By Cauchy's integral theorem we conclude

$$[f] = f_1(t) - f_2(t) = \alpha(t)$$

and similarly

$$g = g_1(t) - g_2(t) = \beta(t) .$$

We introduce these results in (10.28) and obtain after changing the variables of integration:

$$U_1^n = - \frac{1}{(i\omega)^n} \int_0^\infty \alpha\left(\frac{t}{i\omega}\right) t^n \ e^{-t} \ dt$$

(10.29)

$$U_2^n = \frac{1}{(-i\omega)^n} \int_0^\infty \beta\left(\frac{t}{-i\omega}\right) t^n \ e^{-t} \ dt .$$

Since u_1^n and u_2^n are the contributions to U of A_1^n and A_2^n, the contribution of A itself to U or the complete asymptotic series of the function U thus has the form

$$(10.30) \quad U = \sum_{n=0}^{\infty} \left\{ \frac{D_n}{n!} \frac{1}{(-i\omega)^n} \int_0^\infty \beta\left(\frac{t}{-i\omega}\right) t^n \ e^{-t} \ dt - \frac{C_n}{n!} \frac{1}{(i\omega)^n} \int_0^\infty \alpha\left(\frac{t}{i\omega}\right) t^n \ e^{-t} \ dt \right\}$$

if E_o, in the neighborhood of $t = t_o = \frac{1}{c}\psi$, is given by the expressions (10.22). Of course the U so obtained is typical of each of the U_α to be evaluated for the representation (10.12) of the steady state field.

Consider as an example the case that $\alpha = t^k \cdot \beta = t^\sigma$ where k and σ are greater than -1. We obtain readily by use of the Gamma function that

$$(10.31) \qquad U = \sum_{n=0} \frac{1}{n!} \left(\frac{D_n \Gamma(n+1+\sigma)}{(-i\omega)^{n+\sigma}} - \frac{C_n \Gamma(n+1+k)}{(i\omega)^{n+k}} \right) .$$

Consider next the example

$C_n = (-1)^n D_n$ and $\alpha = \beta = \frac{1}{\pi} \log t$. This means that

$$E_o(t) = \frac{1}{\pi} \log|t-t_o| \sum_{n=0}^{\infty} \frac{D_n}{n!} (t-t_o)^n$$

on both sides of $t = t_o$. In this case we obtain from (10.30) that

$$(10.32) \qquad U = i \sum_{n=0}^{\infty} \frac{D_n}{(-i\omega)^n} ,$$

i.e., except for the factor i, the same result as in case

$$E_o = \sum \frac{D_n}{n!} (t-t_o)^n , \quad t > t_o$$

$$= 0 \qquad\qquad , \quad t < t_o .$$

It is apparent from these examples that the result embodied in formula (10.30) can be used to obtain explicit asymptotic expansions in a variety of cases. Further, since U in that formula is typical of the U_α in (10.12) it follows that we can obtain asymptotic expressions for the steady state fields u and v. To obtain terms in the expansions of u, say, of a given order in $\frac{1}{i\omega}$ it would be necessary to collect these terms from the various expressions for the U_α . Formula (10.30) will prove useful in applications thus far made of the above theory. One of these applications, namely, to the asymptotic evaluation of diffraction integrals should be completed shortly.